

CONTINUOUS UNIFORM OR RECTANGULAR DISTRIBUTION

Presented
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Definition:

A continuous random variable X is said to have the uniform distribution over the interval $[a, b]$ if its p.d.f. is given by :

$$\begin{aligned}f(x) &= \frac{1}{b-a}, \quad a \leq x \leq b; -\infty < a < b < \infty \\&= 0 \quad , \text{otherwise}\end{aligned}$$

MOMENTS

$$\mu_r = E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x) dx = \int_a^b x^r \cdot \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[\frac{x^{r+1}}{r+1} \right]_a^b = \frac{1}{b-a} \left[\frac{b^{r+1} - a^{r+1}}{r+1} \right]$$

$$\mu_1 = E(X) = \frac{1}{b-a} \left[\frac{b^2 - a^2}{2} \right] = \frac{a+b}{2}$$

$$\mu_2 = E(X^2) = \frac{1}{b-a} \left[\frac{b^3 - a^3}{3} \right] = \frac{a^2 + ab + b^2}{3}$$

$$Var(X) = \mu_2 - \mu_1^2 = \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2} \right)^2$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12} = \frac{(a-b)^2}{12}$$

$$\mu_{2r} = E[X - E(X)]^{2r} = \int_a^b (x - \frac{a+b}{2})^{2r} \cdot \frac{1}{b-a} \cdot dx$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} (z)^{2r} \cdot \frac{1}{b-a} \cdot dz$$

Let $x - \frac{a+b}{2} = z$
so, $dx = dz$

$$= \frac{2}{b-a} \cdot \int_0^{\frac{b-a}{2}} (z)^{2r} \cdot dz$$

\because the integrand is an even function
in z .

$$= \frac{2}{b-a} \cdot \left[\frac{z^{2r+1}}{2r+1} \right]_0^{\frac{b-a}{2}} = \frac{2}{b-a} \cdot \left[\frac{\left(\frac{b-a}{2}\right)^{2r+1}}{2r+1} \right]$$

$$\mu_2 = E[X - E(X)]^2 = \frac{2}{b-a} \cdot \left[\frac{\left(\frac{b-a}{2}\right)^{2+1}}{2+1} \right] = \frac{(b-a)^2}{12}$$

$$\mu_4 = E[X - E(X)]^4 = \frac{[b-a]^5}{80(b-a)} = \frac{[b-a]^4}{80}$$

$$\mu_{2r+1} = E[X - E(X)]^{2r+1} = \int_a^b (x - \frac{a+b}{2})^{2r+1} \cdot \frac{1}{b-a} \cdot dx$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} (z)^{2r+1} \cdot \frac{1}{b-a} \cdot dz$$

Let $x - \frac{a+b}{2} = z$
so, $dx = dz$

$$= 0$$

\because the integrand is an odd function in z .

The measure of skewness $\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}}$

$= 0$; since, $\mu_3 = 0$.

The measure of kurtosis $\gamma_2 = \beta_2 - 3 = \frac{\mu_4}{\mu_2^2} - 3$

$$= \left[\left[\frac{(b-a)^4}{80} \right] \Big/ \left[\frac{(b-a)^4}{144} \right] \right]^{-3} = -1.2$$

\therefore The distribution is symmetric and platykurtic.

QUARTILE DEVIATION

Cumulative distribution function $F(x) = P(X \leq x)$

$$= \int_a^x \frac{1}{b-a} dt \quad = \frac{x-a}{b-a}$$

$$F(Q_1) = \frac{1}{4} \quad or, \frac{Q_1 - a}{b - a} = \frac{1}{4} \quad or, Q_1 = \frac{b - a}{4} + a = \frac{b + 3a}{4}$$

$$F(Q_3) = \frac{3}{4} \quad or, \frac{Q_3 - a}{b - a} = \frac{3}{4} \quad or, Q_3 = \frac{3(b - a)}{4} + a = \frac{3b + a}{4}$$

$$\text{Quartile deviation} = \frac{Q_3 - Q_1}{2}$$

$$= \left[\frac{3b + a}{4} - \frac{b + 3a}{4} \right] / 2$$

$$= \frac{b - a}{4}$$

$$F(Q_2) = \frac{1}{2} \quad \text{or, } \frac{Q_2 - a}{b - a} = \frac{1}{2}$$

$$\text{or, } Q_2 = \frac{b - a}{2} + a = \frac{b + a}{2}$$

MEAN DEVIATION

$$\text{Mean Deviation} = E[|X - E(X)|]$$

$$= \int_a^b \left| x - \frac{a+b}{2} \right| \cdot \frac{1}{b-a} dx$$

$$= \int_{-\frac{b-a}{2}}^{\frac{b-a}{2}} |z| \cdot \frac{1}{b-a} dz$$

$$= 2 \int_0^{\frac{b-a}{2}} z \cdot \frac{1}{b-a} dz$$
$$= \frac{2}{2(b-a)} \cdot \left(\frac{b-a}{2} \right)^2$$

$$= \frac{b-a}{4}$$

Let $x - \frac{a+b}{2} = z$
so, $dx = dz$

Pr oblem :

For rectangular distribution, find $P(X = \frac{a+b}{2})$.

Solution :

For rectangular distribution,

$$P(X = \frac{a+b}{2}) = \int_{\frac{a+b}{2}}^{\frac{a+b}{2}} \frac{1}{b-a} dx = 0$$

Problem:

A continuous random variable X follows the rectangular distribution over the domain $[1,2]$, find its G.M.

Solution:

$$\begin{aligned} E(\log X) &= \int_1^2 \log x \, dx \\ &= \left[\log x \int dx - \int \left\{ \frac{d}{dx} \log x \int dx \right\} dx \right]_1^2 \\ &= \left[x \cdot \log x - \int dx \right]_1^2 = [x \cdot \log x - x]_1^2 = 2 \log 2 - 1 \\ \therefore \text{Geometric mean} &= \text{anti log}(2 \log 2 - 1) = 0.4 \end{aligned}$$

Problem:

A random variable X has the following p.d.f.

$$f(x) = \begin{cases} k, & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

i) Determine the constant 'k' ii) What is the value of $P(|X| > 1)$?

Solution:

i) Since, $f(x)$ is a p.d.f, so, $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\text{or, } \int_{-2}^2 k dx = 1 \text{ or, } k[x]_{-2}^2 = 1 \text{ or, } k[2 + 2] = 1 \text{ or, } k = \frac{1}{4}$$

$$ii) P(|X| > 1) = P(X < -1) + P(X > 1) = \int_{-\infty}^{-1} \frac{1}{4} dx + \int_1^{\infty} \frac{1}{4} dx$$

$$= \int_{-2}^{-1} \frac{1}{4} dx + \int_1^2 \frac{1}{4} dx$$

$$= \left[\frac{1}{4} \cdot x \right]_{-2}^{-1} + \left[\frac{1}{4} \cdot x \right]_1^2$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Problem: If X is uniformly distributed over $[1,2]$, find k such that

$$P[X > k + E(X)] = \frac{1}{3}$$

Solution:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_1^2 x \cdot \frac{1}{2-1} \cdot dx = \left[\frac{x^2}{2} \right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$P[X > k + E(X)] = \frac{1}{3} \text{ or, } P[X > k + \frac{3}{2}] = \frac{1}{3}$$

$$\text{or, } \int_{k+\frac{3}{2}}^2 f(x) \cdot dx = \frac{1}{3} \text{ or, } \int_{k+\frac{3}{2}}^2 \frac{1}{2-1} \cdot dx = \frac{1}{3} \text{ or, } x \Big|_{k+\frac{3}{2}}^2 = \frac{1}{3}$$

$$\text{or, } 2 - k - \frac{3}{2} = \frac{1}{3} \text{ or, } k = 2 - \frac{3}{2} - \frac{1}{3} = \frac{12 - 9 - 2}{6} = \frac{1}{6}$$



GO AHEAD...