

# **NEGATIVE BINOMIAL DISTRIBUTION**

**Presented  
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## Definition:

A random variable  $X$  is said to have a negative binomial distribution with parameters  $r$  and  $p$  if its p.m.f. is given by:

$$P(X = x) = \binom{x+r-1}{r} \cdot p^r \cdot q^x; x = 0, 1, \dots, 0 < p < 1, q = 1 - p$$
$$= 0 \quad , otherwise$$

$$\begin{aligned}
\text{Also, } \quad & (x+r-1)_{C_{(r-1)}} = (x+r-1)_{C_x} \\
&= \frac{(x+r-1)(x+r-2)\dots\dots(r+1)r}{x!} \\
&= \frac{(-1)^x (-r-x+1)(-r-x+2)\dots\dots(-r-1)(-r)}{x!} \\
&= (-1)^x (-r)_{C_x}
\end{aligned}$$

$$\begin{aligned}
\therefore P(X=x) &= (-r)_{C_x} \cdot p^r \cdot (-q)^x; x=0,1,\dots\dots \\
&= 0 \quad , otherwise
\end{aligned}$$

*Which is the  $(x+1)$ th term in the expansion of  $p^r(1-q)^{-r}$ ,  
a binomial expansion with a (-)ve index. Hence, the  
distribution is known as (-)ve binomial distribution.*

*If  $p = \frac{1}{Q}$  and  $q = \frac{P}{Q}$  so that  $Q - P = 1$  ( $\because p + q = 1$ ) then*

$$\begin{aligned}\therefore P(X = x) &= \binom{-r}{C_x} \cdot Q^{-r} \cdot \left(-\frac{P}{Q}\right)^x; x = 0, 1, \dots \\ &= 0 \quad , \text{otherwise}\end{aligned}$$

*This is the general term in the (-) ve binomial expansion  $(Q - P)^{-r}$*

## POISSON DISTRIBUTION AS A LIMITING CASE OF NEGATIVE BINOMIAL DISTRIBUTION

*Negative binomial distribution tends to Poisson distribution as  $P \rightarrow 0, r \rightarrow \infty$  such that  $rP = \lambda$  (finite).*

$$\begin{aligned} P(X = x) &= (x + r - 1)_{C_{(r-1)}} \cdot p^r \cdot q^x = (x + r - 1)_{C_x} \cdot Q^{-r} \cdot \left(\frac{P}{Q}\right)^x \\ &= \frac{(x + r - 1)(x + r - 2) \dots (r + 1)r}{x!} (1 + P)^{-r} \left(\frac{P}{1 + P}\right)^x \\ &= \frac{1}{x!} \cdot \left(1 + \frac{x-1}{r}\right) \cdot \left(1 + \frac{x-2}{r}\right) \dots \left(1 + \frac{1}{r}\right) \cdot 1 \cdot r^x \cdot (1 + P)^{-r} \left(\frac{P}{1 + P}\right)^x \end{aligned}$$

When,  $r \rightarrow \infty$  then

$$P(X = x) = \frac{1}{x!} \cdot (1 + P)^{-r} \cdot \left(\frac{rP}{1 + P}\right)^x$$

$$= \frac{\lambda^x}{x!} \cdot \left(1 + \frac{\lambda}{r}\right)^{-r} \cdot \left(1 + \frac{\lambda}{r}\right)^{-x}$$

$$\because rP = \lambda$$

$$= \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

when,  $r \rightarrow \infty$  then  $\left(1 + \frac{\lambda}{r}\right)^{-r} = e^{-\lambda}$

and  $\left(1 + \frac{\lambda}{r}\right)^{-x} = 1$

## RECUSION RELATION FOR MOMENTS OF NEGATIVE BINOMIAL DISTRIBUTION

*X is a negative binomial variate with p.m.f*

$$P(X = x) = \binom{k + x - 1}{C_x} \cdot p^k \cdot q^x; x = 0, 1, \dots, 0 < p < 1, q = 1 - p$$
$$= 0 \quad , \text{otherwise}$$

*Show that the moment recurrence formula is*

$$\mu_{r+1} = q \left[ \frac{d\mu_r}{dq} + \frac{rk}{p^2} \mu_{r-1} \right], \quad r = 1, 2, 3, \dots$$

**Proof:**

*For (-)ve binomial distribution with parameters k and p,*

$$\text{mean} = \mu = k \cdot \frac{q}{p}$$

$$\therefore \mu_r = \sum_{x=0}^{\infty} (x - \mu)^r \cdot P(X = x) = \sum_{x=0}^{\infty} \left(x - \frac{k \cdot q}{p}\right)^r \cdot (k + x - 1)_{C_x} \cdot p^k \cdot q^x$$

$$\frac{d\mu_r}{dq} = \sum_{x=0}^{\infty} (k + x - 1)_{C_x} \left[ r \cdot \left(x - \frac{k \cdot q}{p}\right)^{r-1} \cdot \frac{(-k)\{p+q\}}{p^2} \cdot p^k \cdot q^x \right]$$

$$+ \sum_{x=0}^{\infty} (k + x - 1)_{C_x} \cdot \left(x - \frac{k \cdot q}{p}\right)^r \left[ x \cdot p^k \cdot q^{x-1} + q^x \cdot k \cdot p^{k-1} (-1) \right]$$

$$= -\frac{rk}{p^2} \cdot \mu_{r-1} + \sum_{x=0}^{\infty} (x - \frac{k \cdot q}{p})^r \cdot \binom{k+x-1}{C_x} \cdot p^k \cdot q^x \cdot \left\{ \frac{x}{q} - \frac{k}{p} \right\}$$

$$= -\frac{rk}{p^2} \cdot \mu_{r-1} + \frac{1}{q} \sum_{x=0}^{\infty} (x - \frac{k \cdot q}{p})^{r+1} \cdot \binom{k+x-1}{C_x} \cdot p^k \cdot q^x$$

$$= -\frac{rk}{p^2} \cdot \mu_{r-1} + \frac{1}{q} \cdot \mu_{r+1}$$

$$\therefore \mu_{r+1} = q \left[ \frac{d\mu_r}{dq} + \frac{rk}{p^2} \cdot \mu_{r-1} \right], r = 1, 2, 3, \dots$$

**think more**