Paper-Eight
Module-XIV
Full Marks-30
Time- 1.15hr

Answer any TWO questions:

1. a) Show that if $P(A|B) = 1, P(B^c|A^c) = 1$. 3

b) An urn initially contains 5 white and 7 black balls. Each time a ball is selected, its colour is noted and it is replaced in the urn along with 2 other balls of the same colour. Compute the probability that the first 2 balls selected are black and the next 2 white. 3

c) Suppose we have 10 coins such that if the ith coin is flipped, heads will appear with probability $\frac{i}{10}, i = 1,2,\ldots,10$. When one of the coins is randomly selected and flipped, it shows heads. What is the conditional probability that it was the fifth coin? 4

d) Define Poisson trials. What is the probability of obtaining multiple of three twice in a throw with 6 dice? 2+3

2. a) How many people are needed so that the probability that at least one of them has the same birthday as you is greater than $\frac{1}{2}$? 3

b) Suppose $P(X = 0) = 1 - P(X = 1)$. If $E(X) = 3\text{var}(X)$, find $P(X = 0)$. 3

c) The probability density of a random variable X is

$$f(x) = \begin{cases} 2xe^{-x^2}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability density of X^2. 4

d) The random variable X has the p.d.f

$$f(x) = \begin{cases} ax + bx^2, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

If $E(X) = 0.6$, find $a) P\left(X < \frac{1}{2}\right)$ and $b)$ $\text{var}(X)$. 3+2

3. a) Consider a sequence of independent Bernoulli's trials, each of which is a success with probability p. Let X_1 be the no. of failures preceding the first success, and let X_2 be the no. of failures between first two successes. Find the joint mass function of X_1 and X_2. 3

b) The joint p.d.f of X and Y is given by

$$f(x,y) = \begin{cases} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), & 0 < x < 1, 0 < y < 2, \\ 0, & otherwise \end{cases}.$$

Find $P(X > Y), P\left(Y > \frac{1}{2} \middle| X < \frac{1}{2}\right)$. 2+3

c) X is a continuous random variable having spectrum $(-\infty, \infty)$ and distribution function $F(x)$. Show that

$$E(X) = \int_0^\infty \{1 - F(x) - F(-x)\}dx$$

provided $x\{1 - F(x) - F(-x)\} \to 0$ as $x \to \infty$. 4

d) Show that $E\{(X - a)^2\}$ is minimized at $a = E(X)$. 3

4. a) Use central limit theorem to find the approximate probability that the sum obtained is between 30 to 40 when 10 fair dice are rolled.

[Given $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.0184} e^{-\frac{x^2}{4}} dx = 0.846$] 4

b) If X is a standard normal random variable, what is $\mathrm{Cov}(X, X^2)$? 2

c) Derive Poisson distribution as a limit of Binomial distribution. 4

d) If X is a non-negative random variables having mean m, prove that $P(X \geq \tau m) \leq \frac{1}{\tau}$ for any $\tau > 0$. 5