Lecture Note on Partial Differential Equation

Portions covered:

- **The Cauchy Problem**
- **Cauchy-Kowalewskaya Theorem**
- **Cauchy Problem of finite and infinite string**
- **Initial boundary value problems**
- **Semi-infinite string with a fixed end**
- **Semi-infinite string with a free end**
- **Equations with non-homogeneous boundary conditions**
- **Non-homogeneous wave equation**
- **Method of separation of variables**
- **Solving vibrating string problem**
- **Solving the heat conduction problem**

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The Cauchy Problem:

Consider the 2nd order linear PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H\left(x, y, u, u_x, u_y\right) \dots (1)$$

where the coefficients A, B, C are functions of x and y.

Let (x_0, y_0) be a point on a smooth curve L_0 in the xy −plane. Let the curve has the parametric form

$$x_0 = x_0(\lambda), y_0 = y_0(\lambda), \dots (2)$$

λ being the parameter.

The Cauchy problem is to determine the value of $u(x, y)$ in the neighbourhood of the curve L_0 where the following conditions are satisfied:

$$u = f(\lambda) \text{ on } L_0 \dots (3a)$$

$$\frac{\partial u}{\partial n} = g(\lambda) \text{ on } L_0 \dots (3b)$$

$\hat{n}$ is the unit normal to L_0 which lies to the left of L_0 while L_0 is traced in counter clock-wise direction. The functions f and g, prescribed on L_0, are called Cauchy data.

For every point on L_0, the value of u is specified by $(3a)$. The curve L_0 in the xy −plane alongwith the condition $(3a)$ defines a curve L in (x, y, u) −space whose projection on

xy −plane is L_0. The solution of Cauchy problem is thus an integral surface in (x, y, u) −space which passes through L and satisfies the condition $(3b)$.

Let us suppose that the function $f(\lambda)$ is continuously differentiable. Then along L_0,

$$\frac{du}{d\lambda} = \frac{\partial u}{\partial x}\frac{dx}{d\lambda} + \frac{\partial u}{\partial y}\frac{dy}{d\lambda} = \frac{df}{d\lambda}, \ldots (4a)$$

$$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial x}\frac{dy}{ds} + \frac{\partial u}{\partial y}\frac{dx}{ds} = g \ldots (4b)$$

$[\because \frac{\partial u}{\partial n} = \vec{\nabla}u.\hat{n} = g, \hat{n} = \left(-\frac{dy}{ds}, \frac{dx}{ds}\right)]$

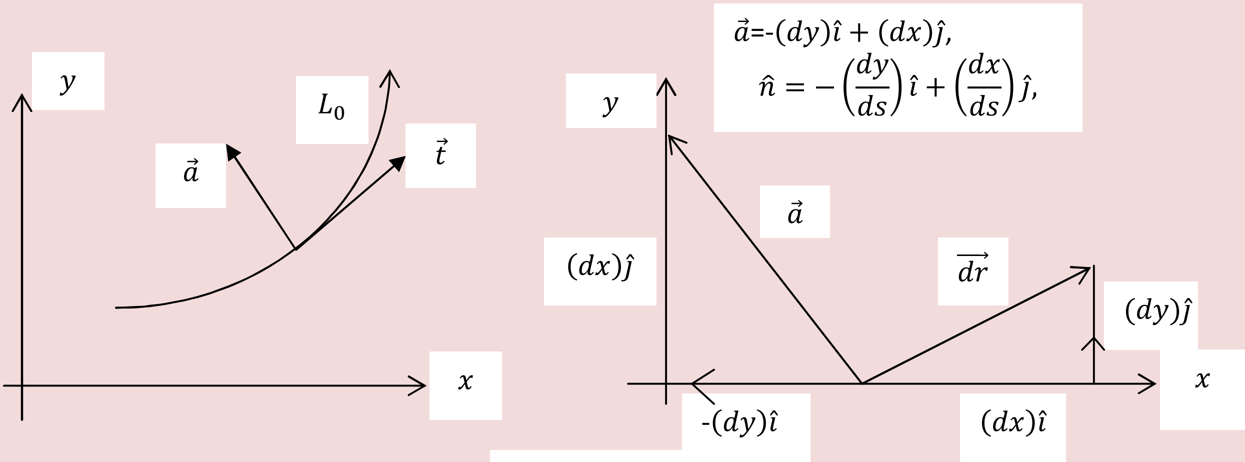


Fig. 1

Since, $\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ -\frac{dy}{ds} & \frac{dx}{ds} \end{vmatrix} = \frac{(dx)^2+(dy)^2}{(ds)(d\lambda)} \neq 0$, the equations $(4a)$ and $(4b)$ are solvable for u_x and u_y.

Therefore, u_x and u_y are now known on L_0. Differentiating u_x and u_y w.r.t λ, we find

$$\frac{d}{d\lambda}(u_x) = u_{xx}\frac{dx}{d\lambda} + u_{xy}\frac{dy}{d\lambda}, \ldots (5a)$$

$$\frac{d}{d\lambda}(u_y) = u_{xy}\frac{dx}{d\lambda} + u_{yy}\frac{dy}{d\lambda} \ldots . (5b)$$

Again, we have the PDE

$$Au_{xx} + Bu_{xy} + Cu_{yy} = H(x, y, u, u_x, u_y) \ldots (6)$$

Now, H is prescribed on L_0 as u, u_x, u_y are all given on L_0. If

$$\begin{vmatrix} \frac{dx}{d\lambda} & \frac{dy}{d\lambda} & 0 \\ 0 & \frac{dx}{d\lambda} & \frac{dy}{d\lambda} \\ A & B & C \end{vmatrix} \neq 0, or\ C\left(\frac{dx}{d\lambda}\right)^2 - B\left(\frac{dx}{d\lambda}\right)\left(\frac{dy}{d\lambda}\right) + A\left(\frac{dy}{d\lambda}\right)^2 \neq 0,$$

$$or\ A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C \neq 0, i.e. L_0 \text{ must not coincide with characteristic curves,}$$

then u_{xx}, u_{xy}, u_{yy} can be solved uniquely from $(5a)$, $(5b)$ and (6).

Hence, u_{xx}, u_{xy}, u_{yy} are obtained on L_0. Now, if A, B, C, f and g are analytic functions then all higher derivatives of $u(x, y)$ can be computed by the above procedure. This produces the Taylor's series of $u(x, y)$ about (x_0, y_0)

$$u(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!\,(n-k)!}\left(\frac{\partial^n u_0}{\partial x_0^k \partial y_0^{n-k}}\right), \qquad u_0 = u(x_0, y_0).$$

The Cauchy-Kowalewskaya Theorem:

Let the PDE is given in the form

$$u_{yy} = \sum_{i,j=1}^{n} a_{ij}(y, x_1, x_2, \dots, x_n)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_{0i}(y, x_1, x_2, \dots, x_n)\frac{\partial^2 u}{\partial y \partial x_i}$$
$$+ \sum_{i=1}^{n} b_i(y, x_1, x_2, \dots, x_n)\frac{\partial u}{\partial x_i} + b_0(y, x_1, x_2, \dots, x_n)\frac{\partial u}{\partial y} + c(y, x_1, x_2, \dots, x_n)u$$
$$+ h(y, x_1, x_2, \dots, x_n) \dots (8)$$

Here, $a_{ij}, a_{0i}, b_i, b_0, c, h$ are all analytic functions of their arguments in the neighbourhood of point $(y^0, x_1^0, x_2^0, \dots, x_n^0)$.

The Cauchy problem for this equation consists in finding a solution satisfying the initial conditions

$$u(y, x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \text{ on } y = y^0,$$

$$u_y(y, x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) \text{ on } y = y^0$$

where f and g are analytic in the neighbourhood of $(x_1^0, x_2^0, \dots, x_n^0)$. Then the Cauchy problem has an unique solution in some neighbourhood of $(y^0, x_1^0, x_2^0, \dots, x_n^0)$.

Wave Equation:

Two-dimensional wave equation:

Let us consider a flexible and stretched string under the tension T attached between two fixed end points at a distance l apart.

The following assumptions are made in order to obtain a simple equation:

- The string is made of homogeneous material.
- The tension of the string is constant.
- Only transverse vibration takes place.
- The deflection is small compared to the wave length of the string.
- The slope of deflected string at any point is small compared with unity.

Let the two fixed ends of the string be attached at $O(0,0)$ and $L(l,0)$. The string is assumed to lie on the $x-$axis in the equilibrium position of the string. We consider a differential element PQ of the string. Let T be the tension of the string at P and Q as shown in fig. 2. If a vibration is made in the $xu-$plane, the displacement u of the string at a time t will be a function of x and t. Let us further assume that the string length δx is stretched to δs.

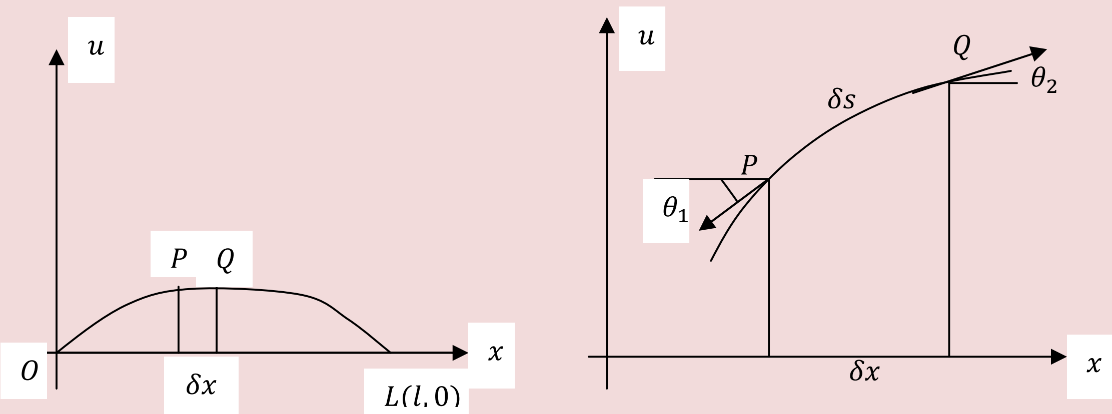


Fig. 2

The force acting on the element of the string in the vertical direction is $T\sin\theta_2 - T\sin\theta_1$.

Due to Newton's law of motion, the resultant force is equal to the mass times the acceleration.

$$\therefore T\sin\theta_2 - T\sin\theta_1 = (\rho\delta x)u_{tt}(x,t) \ldots (1)$$

where ρ is the mass per unit length of the string in equilibrium position.

Since, the angles α and β are very small, $\sin\theta_1 \cong \tan\theta_1$, $\sin\theta_2 \cong \tan\theta_2$.

From (1), $T\tan\theta_2 - T\tan\theta_1 = (\rho\delta x)u_{tt}(x,t), or\ \tan\theta_2 - \tan\theta_1 = \frac{\rho\delta x}{T}u_{tt}(x,t) \ldots (2)$

Now, $\tan\theta_1$ and $\tan\theta_2$ are the slopes of the string at x and at $x+\delta x$.

$$\therefore \tan\theta_1 = u_x(x,t), \qquad \tan\theta_2 = u_x(x+\delta x, t).$$

$$\therefore u_x(x+\delta x,t) - u_x(x,t) = \frac{\rho\delta x}{T} u_{tt}(x,t),$$

$$\frac{u_x(x+\delta x,t) - u_x(x,t)}{\delta x} = \frac{\rho}{T} u_{tt}(x,t),$$

Taking limit as $\delta x \to 0$,

$$c^2 u_{xx} = u_{tt}(x,t), c^2 = \frac{T}{\rho}.$$

If there is any additional force f acting per unit length of the string, then

$$T u_{xx}(x,t) + f = \rho u_{tt}(x,t),$$

$$or, c^2 u_{xx} + F = u_{tt}(x,t), F = \frac{f}{\rho}.$$

<u>Solution of one dimensional wave equation by Canonical reduction:</u>

Consider the wave equation (Homogeneous equation)

$$u_{tt} = c^2 u_{xx}$$

The characteristic equations are

$$\frac{dx}{dt} = \frac{\pm\sqrt{4c^2}}{2} = \pm c.$$

The characteristic curves are given by $\xi = x + ct, \eta = x - ct$.

$$\therefore u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta, u_{xx} = \left(u_\xi + u_\eta\right)_\xi \xi_x + \left(u_\xi + u_\eta\right)_\eta \eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta},$$

$$u_t = u_\xi \xi_t + u_\eta \eta_t = c\left(u_\xi - u_\eta\right), u_{tt} = c\left(u_\xi - u_\eta\right)_\xi \xi_t + c\left(u_\xi - u_\eta\right)_\eta \eta_t$$
$$= c^2 u_{\xi\xi} - 2c^2 u_{\xi\eta} + c^2 u_{\eta\eta}.$$

Substituting the expression of u_{tt} and u_{xx} in the PDE,

$$4c^2 u_{\xi\eta} = 0, or\ u_{\xi\eta} = 0 \ldots (2)$$

Integrating (2) w.r.t η, we find

$$u_\xi = \psi^*(\xi).$$

Again, integrating w.r.t ξ, $u(\xi,\eta) = \int \psi^*(\xi)d\,\xi + \varphi(\eta)$

$$= \psi(\xi) + \varphi(\eta)\ [\psi, \varphi \text{ are arbitrary } f\text{unctions}]$$

$$= \psi(x + ct) + \varphi(x - ct) \ldots (3)$$

[A wave motion is generally represented by equation of the form $y = f(x - ct) \ldots (A)$ considering the axis of x to be horizontal and the axis of y to be vertically upwards.

y represents the position of a particle at x at time t. If t is increased by T, and x is increased by cT, then $f\big(x + cT - c(t + T)\big) = f(x - ct) = y$. This shows that the wave profile $y = f(x)$ moves with velocity c in the positive x −direction (cT distance is covered in time T, so c is traversed in unit time).

Similarly, the profile $f(x + ct)$ represents a progressive wave travelling to negative x −direction with velocity c.]

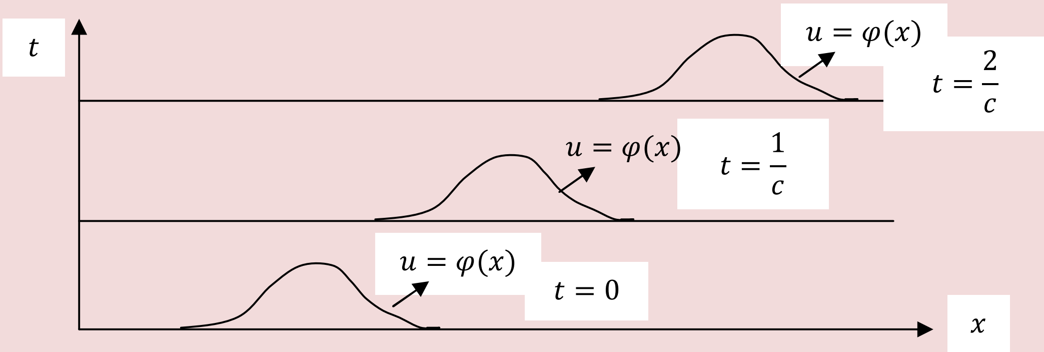


Fig. 3

$\psi(x + ct)$ represents a progressive wave profile travelling to negative x −direction with speed c without change of shape. Again, $\varphi(x - ct)$ is also a progressive wave profile travelling to positive x −direction with speed c without change in shape. $\varphi(x - ct)$ is a function of x for a given time t. At $t = 0$, the shape of the function is given by $\varphi(x)$. At any time t, the shape of function is $\varphi(x - ct)$ or $\varphi(\eta)$ where $\eta = x - ct$ is the new coordinate system obtained by translating the origin at a distance ct to the right. Thus, the shape of the curve remains the same as time progresses and the profile moves to the right with velocity c.

<u>The initial value problem- D' Alembert's Solution:</u>

Consider the PDE

$$u_{tt} = c^2 u_{xx}, x \in R, t > 0,$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad x \in R. \text{[Initial Conditions]} \ldots (4)$$

We have already obtained

$$u(x,t) = \psi(x+ct) + \varphi(x-ct) \dots (5)$$

Hence, the general solution (5) of one-dimensional wave equation represents the super position of two arbitrary wave profiles, both of which are travelling with common speed c but in the opposite directions along x −axis.

Applying the I.Cs (4),

$$u(x,0) = \psi(x) + \varphi(x) = f(x) \dots (6),$$

$$u_t(x,0) = c\psi'_x - c\varphi'_x = g(x) \dots (7).$$

Integration of (7) gives,

$$\psi(x) - \varphi(x) = \frac{1}{c}\int_{x_0}^{x} g(\tau)d\tau + D, \dots (8)$$

x_0 and D being the arbitrary constants.

Solving (6) and (8),

$$\psi(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_{x_0}^{x} g(\tau)d\tau + \frac{D}{2},$$

$$\varphi(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_{x_0}^{x} g(\tau)d\tau - \frac{D}{2}.$$

The solution is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\left[\int_{x_0}^{x+ct} g(\tau)d\tau - \int_{x_0}^{x-ct} g(\tau)d\tau\right]$$

$$= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau \dots (9)$$

Equation (9) is called the D'Alembert's solution of the Cauchy problem for one-dimensional wave equation.

Note:

- The solution $u(x,t)$ depends only on the initial values of f at points $x+ct$ and $x-ct$ and the values of g between the two values. In other words, the solution does not depend on all values outside the interval $x-ct \le x \le x+ct$. This interval is called the domain of dependence of the variable (x,t).

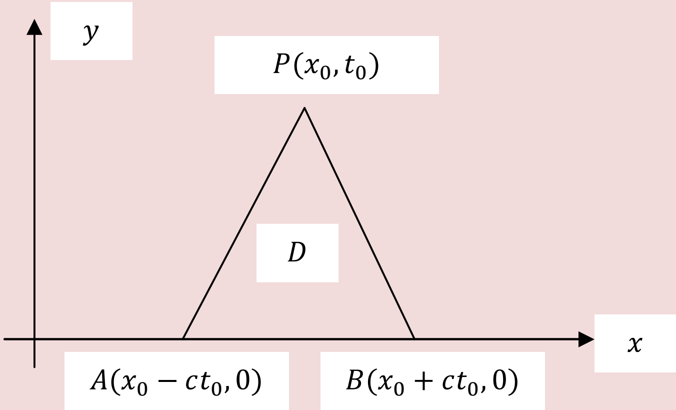


Fig. 4

- According to (9), the value of $u(x_0, t_0)$ depends on the initial data f and g and on the interval $[x_0 - ct_0, x_0 + ct_0]$ which is cut out from the initial line by the two characteristic curves $x \pm ct =$ const having slopes $\left(\pm\frac{1}{c}\right)$ passing through (x_0, t_0). The interval $[x_0 - ct_0, x_0 + ct_0]$ on the line $t = 0$ is called domain of dependence of the solution at the point (x_0, t_0).
- The solution $u(x,t)$ at every point (x,t) inside the triangle D is completely determined by the Cauchy data on the interval $[x_0 - ct_0, x_0 + ct_0]$ (cf. fig. 4). The region D is called region of determinacy of the solution.

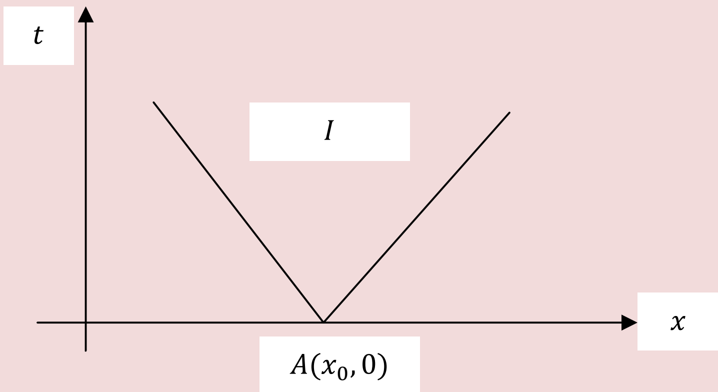


Fig. 5

- The disturbance at a point $(x_0, 0)$ on the x –axis influences the value of $u(x,t)$ in the wedge-shape region $I(x_0) = \{(x,t)|x_0 - ct \leq x \leq x_0 + ct\}$. I is called the region of influence of the point $(x_0, 0)$(cf. fig. 5).

If $g(x) = 0$, i.e. the string is released from rest, then

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)].$$

For example (cf. Sneddon [1]) let us take

$$f(x) = \begin{cases} 0, & x < -a, \\ 1, & |x| < a, \\ 0, & x > a. \end{cases}$$

The motion may be then represented by a series of graphs corresponding to different values of time (cf. fig 5).

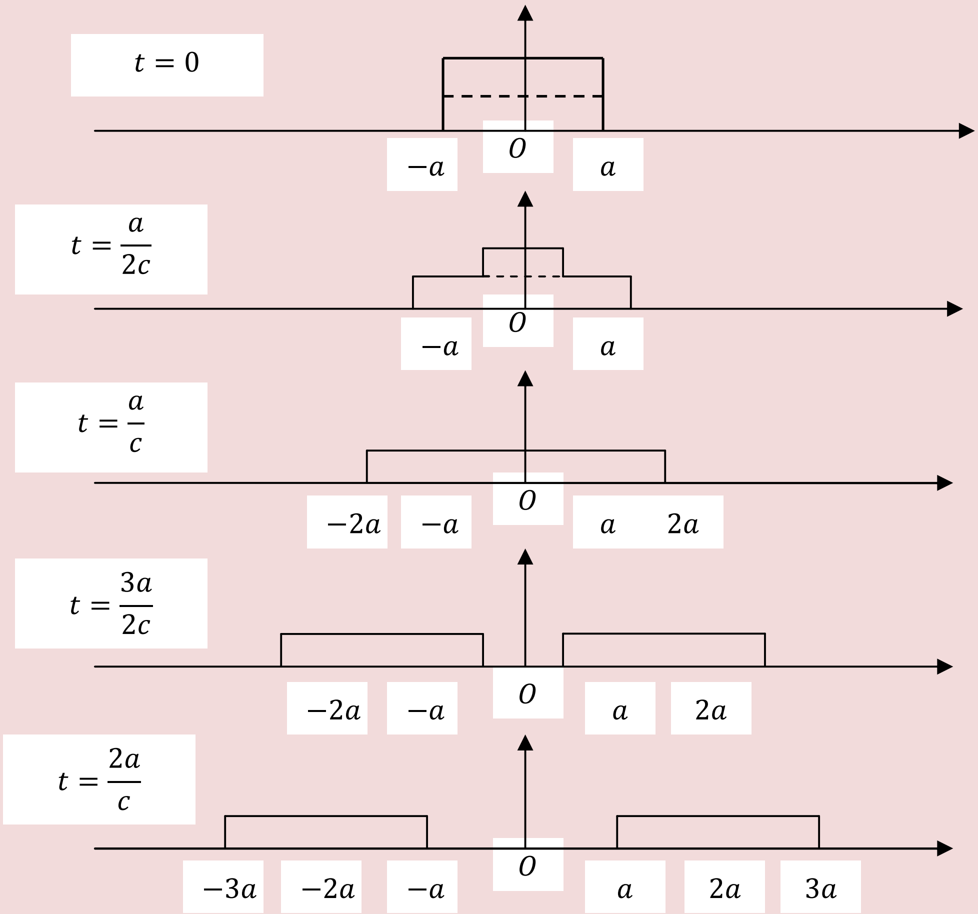


Fig. 5

<u>Initial-boundary value problem:</u>

Semi-infinite string with a fixed end:

Let us consider a semi-infinite vibrating string with a fixed end. Consider the PDE

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \qquad t > 0,$$

$$\left.\begin{aligned} u(x,0) &= f(x), \qquad 0 \le x < \infty \\ u_t(x,0) &= g(x), \qquad 0 \le x < \infty \end{aligned}\right\} \longrightarrow \text{I.Cs}$$

$$u(0,t) = 0, \qquad t > 0. \longrightarrow \text{B.C (Homogeneous)}$$

Here, the initial displacement is always zero.

For $x > ct$, the solution is same as D'Alembert's solution of infinite string. The displacement is influenced by the initial data on $[x - ct, x + ct]$.

For $x < ct$, the interval $[x - ct, x + ct]$ extends to the negative side of the x −axis where the functions f and g are not prescribed.

D' Alembert's formula yields

$$u(x,t) = \psi(x + ct) + \varphi(x - ct) \ldots (10)$$

where

$$\left.\begin{aligned}\psi(\xi) &= \tfrac{1}{2}f(\xi) + \tfrac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \tfrac{D}{2}, \xi \geq 0,\\ \varphi(\eta) &= \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{D}{2}, \eta \geq 0.\end{aligned}\right\} \ldots (11)$$

Now, we apply the boundary condition.

$$u(0,t) = 0 = \psi(ct) + \varphi(-ct),$$

$$or, \varphi(-ct) = -\psi(ct).$$

Letting $\alpha = -ct$, we find, $\varphi(\alpha) = -\psi(-\alpha), \alpha \leq 0$.

Replacing α by $x - ct$, $\varphi(x - ct) = -\psi(ct - x)$.

$$\therefore \varphi(x - ct) = -\frac{1}{2}f(ct - x) - \frac{1}{2c}\int_0^{ct-x} g(\tau)d\tau - \frac{D}{2}.$$

The solution is thus given by

$$u(x,t) = \begin{cases} \dfrac{1}{2}[f(x + ct) + f(x - ct)] + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\tau)d\tau, & x > ct \\ \dfrac{1}{2}[f(x + ct) - f(ct - x)] + \dfrac{1}{2c}\displaystyle\int_{ct-x}^{x+ct} g(\tau)d\tau, & x < ct. \end{cases} \ldots (12)$$

For the existence of the solution u, the function f should be twice continuously differentiable and g to be continuously differentiable.

For the solution $u(x,t)$ to be continuous,

$$u(ct + 0, t) = u(ct - 0, t),$$

$$or, \frac{1}{2}[f(2ct) + f(0)] + \frac{1}{2c}\int_0^{2ct} g(\tau)d\tau = \frac{1}{2}[f(2ct) - f(0)] + \frac{1}{2c}\int_0^{2ct} g(\tau)d\tau$$

$$or, f(0) = 0.$$

Again, for u_x, u_t, u_{xx}, u_{tt} to be continuous at $x = ct$,

$$u_x(ct+0,t) - u_x(ct-0,t) = -\frac{1}{2c}[g(0)+g(0)] = -\frac{1}{c}g(0) = 0.$$

$$u_t(ct+0,t) - u_t(ct-0,t) = \frac{c}{2c}[g(0)+g(0)] = g(0) = 0,$$

$$u_{xx}(ct+0,t) - u_{xx}(ct-0,t) = \frac{1}{2}[f''(0+f''(0)] = f''(0) = 0,$$

$$u_{tt}(ct+0,t) - u_{tt}(ct-0,t) = \frac{c^2}{2}f''(0+f''(0)] = f''(0) = 0.$$

$$\therefore f(0) = g(0) = f''(0) = 0.$$

For example, let us take the problem (cf. Stavroulakis etal. [5]),

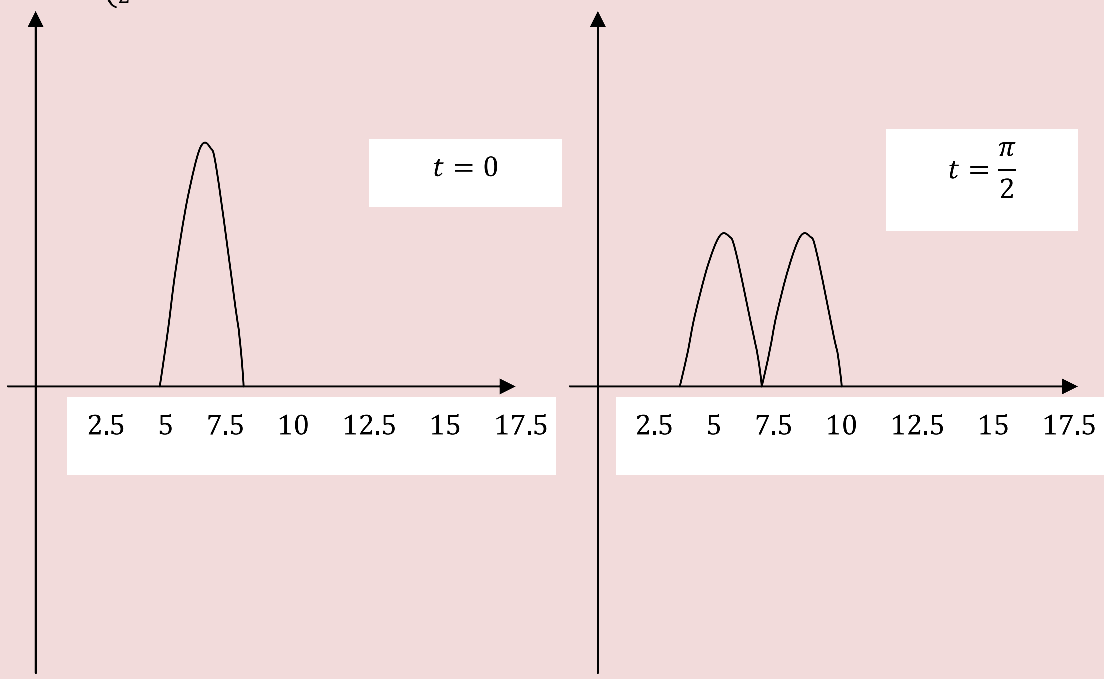
$$u_{tt} = u_{xx}, \qquad 0 < x < \infty, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 \le x < \infty, \qquad f(x) = \begin{cases} \cos^3 x, & \frac{3\pi}{2} < x < \frac{5\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

$$u_t(x,0) = 0 \qquad 0 \le x < \infty, \qquad u(0,t) = 0, \ \ t \ge 0.$$

The solution is thus given by

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+t)+f(x-t)], & x > t \\ \frac{1}{2}[f(x+t)-f(t-x)], & x < t. \end{cases}$$



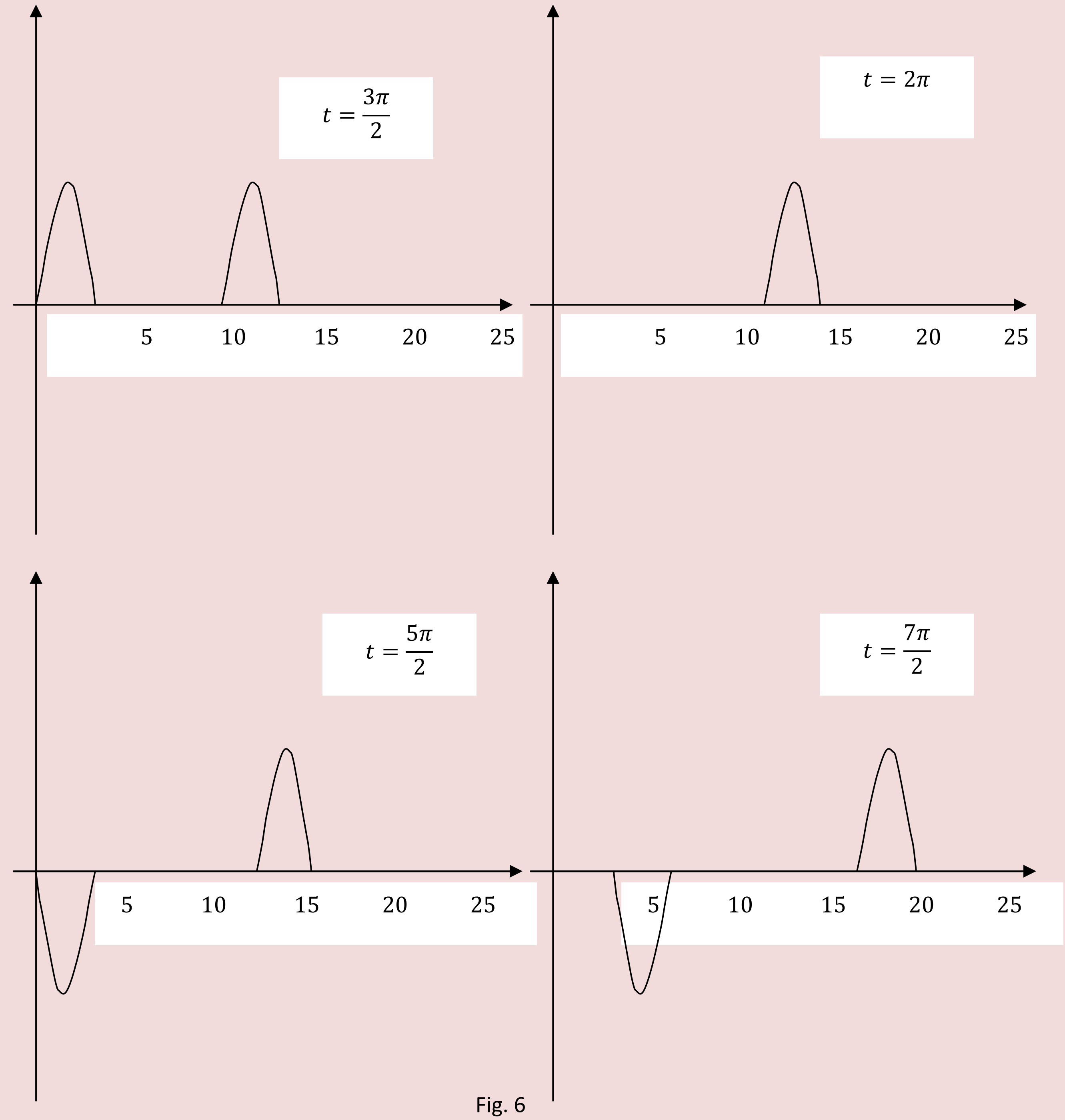


Fig. 6

Semi-Infinite string with a free end:

Homogeneous boundary condition

Consider the PDE

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < \infty, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 \le x < \infty,$$

$$u_t(x,0) = g(x), \qquad 0 \le x < \infty,$$

$$u_x(0,t) = 0, \qquad 0 \le t < \infty.$$

For $x > ct$, the solution is same as D'Alembert's solution and is given by equ. $(10)$and (11).

D' Alembert's solution gives

$$u(x,t) = \psi(x+ct) + \varphi(x-ct),$$

$$u_x(x,t) = \psi'(x+ct) + \varphi'(x-ct).$$

Now, we apply the B.C $u_x(0,t) = 0$. This produces

$$\psi'(ct) + \varphi'(-ct) = 0 \quad \dots (13)$$

On integration of (13) w.r.t t,

$$\psi(ct) - \varphi(-ct) = D, D \text{ is integration constant.}$$

Letting $\alpha = -ct$, $\varphi(\alpha) = \psi(-\alpha) - D, \alpha \le 0$.

Replacing now α by $x - ct$, we find

$$\varphi(x-ct) = \psi(ct-x) - D = \frac{1}{2} f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(\tau)d\tau - \frac{D}{2}.$$

Therefore,

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, & x > ct \\ \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)d\tau + \int_0^{ct-x} g(\tau)d\tau\right], & x < ct. \end{cases} \quad \dots (14)$$

For the solution to exists, f must be twice continuously differentiable and g must be continuously differentiable.

It is clear from (14) that $u(x,t)$ is continuous at $x = ct$.

The continuity of u_x, u_t, u_{xx}, u_{tt} at $x = ct$ implies the following conditions:

$$u_x(ct+0,t) - u_x(ct-0,t) = \frac{1}{2}[f'(0) + f'(0)] = f'(0) = 0,$$

$$u_t(ct+0,t) - u_t(ct-0,t) = \frac{c}{2}[-f'(0) - f'(0)] = -cf'(0) = 0,$$

$$u_{xx}(ct+0,t) - u_{xx}(ct-0,t) = -\frac{1}{2c}[g'(0) + g'(0)] = -\frac{1}{c} g'(0) = 0,$$

$$u_{tt}(ct+0,t)-u_{tt}(ct-0,t)=-\frac{c^2}{2c}[g'(0)+g'(0)]=-cg'(0)=0.$$

$$\therefore f'(0)=g'(0)=0.$$

Non-homogeneous boundary condition

i) Consider the problem

$$u_{tt}=c^2u_{xx}, \qquad 0<x<\infty, \qquad t>0,$$

$$u(x,0)=f(x), \qquad 0\le x<\infty,$$

$$u_t(x,0)=g(x), \qquad 0\le x<\infty,$$

$$u(0,t)=p(t), \qquad t\ge 0.$$

For $x>ct$, the solution is given by equ. $(10)$and (11).

Using the boundary condition on D'Alembert's solution we find,

$$u(0,t)=p(t)=\psi(ct)+\varphi(-ct),$$

$$or, \varphi(-ct)=p(t)-\psi(ct).$$

Letting $\alpha=-ct$, we find, $\varphi(\alpha)=p\left(-\frac{\alpha}{c}\right)-\psi(-\alpha), \alpha\le 0.$

Replacing α by $x-ct$, $\varphi(x-ct)=p\left(t-\frac{x}{c}\right)-\psi(ct-x).$

$$u(x,t)=\begin{cases}\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\tau)d\tau, & x>ct\\ \frac{1}{2}[f(x+ct)-f(ct-x)]+\frac{1}{2c}\int_{ct-x}^{x+ct}g(\tau)d\tau+p\left(t-\frac{x}{c}\right), & x<ct.\end{cases} \quad \dots(15)$$

For $u, u_x, u_t, u_{xx}, u_{tt}$ to be continuous at $x=ct$, the following conditions are to be satisfied:

$$u(ct+0,t)-u(ct-0,t)=\frac{1}{2}[f(0+f(0)]-p(t-t)=f(0)-p(0)=0,$$

$$u_x(ct+0,t)-u_x(ct-0,t)=-\frac{1}{2c}[g(0)+g(0)]+\frac{1}{c}p'(0)=-\frac{1}{c}g(0)+\frac{1}{c}p'(0)=0,$$

$$u_t(ct+0,t)-u_t(ct-0,t)=\frac{1}{2c}[g(0)+g(0)]-\frac{1}{c}p'(0)=\frac{1}{c}g(0)-\frac{1}{c}p'(0)=0,$$

$$u_{xx}(ct+0,t)-u_{xx}(ct-0,t)=\frac{1}{2}[f''(0)+f''(0)]-\frac{1}{c^2}p''(0)=f''(0)-\frac{1}{c^2}p''(0)=0,$$

$$u_{tt}(ct+0,t)-u_{tt}(ct-0,t)=\frac{c^2}{2}[f''(0)+f''(0)]-p''(0)=0.$$

$$\therefore f(0)=p(0), \qquad g(0)=p'(0), \qquad f''(0)=\frac{1}{c^2}p''(0).$$

$ii)$ Now, consider the problem

$$u_{tt}=c^2u_{xx}, \qquad 0<x<\infty, t>0,$$

$$u(x,0)=f(x), \qquad 0\le x<\infty,$$

$$u_t(x,0)=g(x), \qquad 0\le x<\infty,$$

$$u_x(0,t)=q(t), \qquad t\ge 0.$$

For $x>ct$, the solution is given by equ. $(10)$and (11).

On application of B.C on D'Alembert's solution, we find

$$u_x(0,t)=q(t)=\psi'(ct)+\varphi'(-ct) \dots (16)$$

Integrating (16) w.r.t t,

$$\psi(ct)-\varphi(-ct)=c\int_0^t q(\tau)d\tau+D.$$

Letting $\alpha=-ct, \psi(-\alpha)-\varphi(\alpha)=c\int_0^{-\frac{\alpha}{c}}q(\tau)d\tau+D,$

$$\varphi(\alpha)=\psi(-\alpha)-c\int_0^{-\frac{\alpha}{c}} q(\tau)d\tau-D, \alpha\le 0.$$

Replacing α by $x-ct$, we find

$$\varphi(x-ct)=\psi(ct-x)-c\int_0^{t-\frac{x}{c}} q(\tau)d\tau-D.$$

$$u(x,t)=\begin{cases}\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\tau)d\tau, & x>ct\\ \frac{1}{2}[f(x+ct)+f(ct-x)]+\frac{1}{2c}\left[\int_0^{x+ct}g(\tau)d\tau+\int_0^{ct-x}g(\tau)d\tau\right]-c\int_0^{t-\frac{x}{c}}q(\tau)d\tau, & x<ct.\end{cases} \dots (17)$$

For the solution to exist, the function f must be twice continuously differentiable and g must be differentiable.

It is noted that the function $u(x,t)$ is continuous at $x = ct$.

For u_x, u_t, u_{xx}, u_{tt} to be continuous at $x = ct$, the following conditions must hold:

$$u_x(ct+0,t) - u_x(ct-0,t) = \frac{1}{2}[f'(0) + f'(0)] - q(0) = f'(0) - q(0) = 0,$$

$$u_t(ct+0,t) - u_t(ct-0,t) = \frac{c}{2}[-f'(0) - f'(0)] + cq(0) = -cf'(0) + cq(0) = 0,$$

$$u_{xx}(ct+0,t) - u_{xx}(ct-0,t) = -\frac{1}{2c}[g'(0) + g'(0)] + \frac{1}{c}q'(0) = -\frac{1}{c}g'(0) + \frac{1}{c}q'(0) = 0,$$

$$u_{tt}(ct+0,t) - u_{tt}(ct-0,t) = -\frac{c^2}{2c}[g'(0) + g'(0)] + cq'(0) = -cg'(0) + cq'(0) = 0.$$

$$\therefore f'(0) = q(0), \qquad g'(0) = q'(0).$$

<u>Vibrating finite string with fixed end:</u>

Homogeneous boundary conditions:

Let us consider a vibrating string of length l fixed at both ends. The boundary value problem is

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$u(x,0) = f(x), \qquad 0 \le x < l,$$

$$u_t(x,0) = g(x), \qquad 0 \le x < l,$$

$$u(0,t) = 0, u(l,t) = 0, \qquad t \ge 0.$$

The D' Alembert's solution of the wave equation is

$$u(x,t) = \psi(x+ct) + \varphi(x-ct). \dots (18)$$

where

$$\left.\begin{aligned} \psi(\xi) &= \tfrac{1}{2}f(\xi) + \tfrac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \tfrac{D}{2}, \qquad 0 \le \xi \le l, \\ \varphi(\eta) &= \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{D}{2}, \qquad 0 \le \eta \le l. \end{aligned}\right\} \dots (19)$$

Therefore,

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}g(\tau)d\tau\,, \dots(20)$$

$0\le x+ct\le l$ and $0\le x-ct\le l$ [Since functions f and g are defined only over the interval $[0,l]$].

The solution of the given problem is thus uniquely determined only for $0\le t\le \min\left\{\frac{l-x}{c},\frac{x}{c}\right\}$, $x\le l$.

We need to find solution for $x+ct>l$ and $x<ct$.

Applying the B.Cs $u(0,t)=0=\ u(l,t)$, we find

$$u(0,t)=\psi(ct)+\varphi(-ct)=0, t\ge 0,$$

$$u(l,t)=\psi(l+ct)+\varphi(l-ct)=0, t\ge 0.$$

$$\therefore \psi(ct)=-\varphi(-ct),\qquad \psi(l+ct)=-\varphi(l-ct), t\ge 0.$$

If we let $\alpha=-ct, \varphi(\alpha)=-\psi(-\alpha),\ \ \alpha\le 0.\ \ \dots(21)$

If we let $\alpha=l+ct, \psi(\alpha)=-\varphi(2l-\alpha),\ \ \alpha\ge l.\dots(22)$

Replacing α by η in (21), we obtain

$$\varphi(\eta)=-\psi(-\eta)=-\frac{1}{2}\,f(-\eta)-\frac{1}{2c}\int_0^{-\eta}g(\tau)d\tau-\frac{D}{2}, 0\le -\eta\le l, or -l\le\eta\le 0\dots(23)$$

Now, replacing α by ξ in (21), we obtain

$$\psi(\xi)=-\varphi(2l-\xi)=-\frac{1}{2}\,f(2l-\xi)+\frac{1}{2c}\int_0^{2l-\xi}g(\tau)d\tau+\frac{D}{2}, 0\le 2l-\xi\le l,$$

$$or\ \ l\le\xi\le 2l.\dots(24)$$

Therefore, the range of ψ and φ is now extended to $l\le\xi\le 2l$ and $-l\le\eta\le 0$ respectively.

For $-2l\le\eta\le -l$,

$$\varphi(\eta)=-\psi(-\eta)=\frac{1}{2}\,f(2l+\eta)-\frac{1}{2c}\int_0^{2l+\eta}g(\tau)d\tau-\frac{D}{2},$$

and for $2l\le\xi\le 3l$,

$$\psi(\xi)=-\varphi(2l-\xi)=\frac{1}{2}\,f(\xi-2l)+\frac{1}{2c}\int_0^{\xi-2l}g(\tau)d\tau+\frac{D}{2}.$$

Hence, range of ψ and φ is extended to $2l\le\xi\le 3l$ and $-2l\le\eta\le -l$ respectively.

Continuing in this way, $\psi(\xi)$ can be defined for all $\xi \geq 0$ and $\varphi(\eta)$ can be defined for all $\eta \leq 0$.

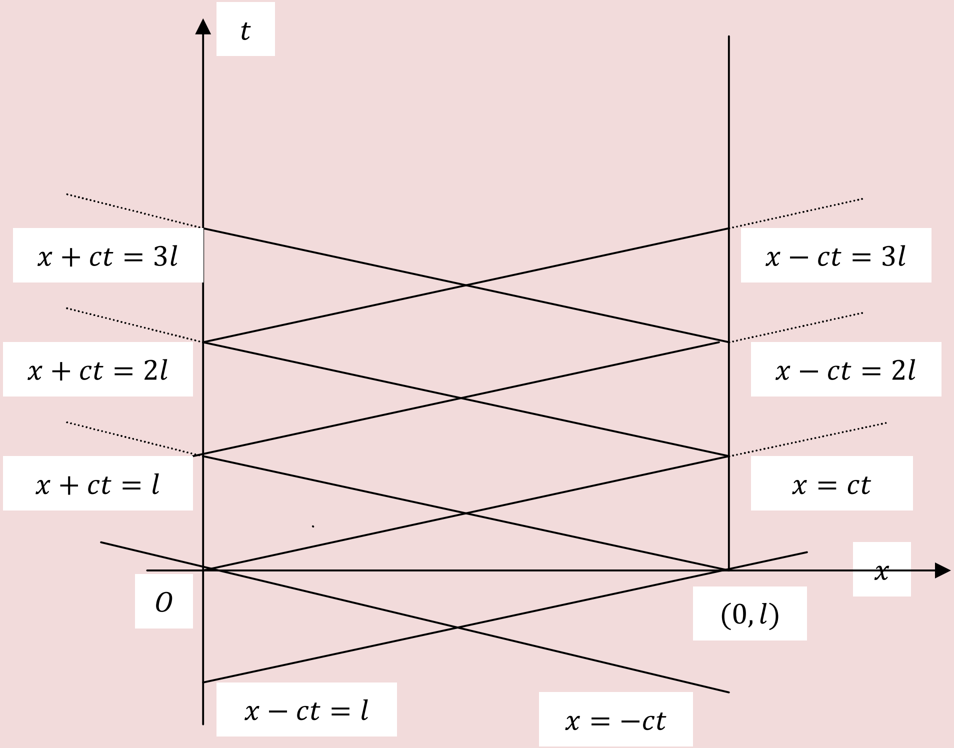


Fig. 7

Non- Homogeneous boundary conditions:

Consider the problem

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < l, t > 0,$$

$$u(x,0) = f(x), \qquad 0 \leq x < l,$$

$$u_t(x,0) = g(x), \qquad 0 \leq x < l,$$

$$u(0,t) = p(t), \qquad u(l,t) = q(t), \qquad t \geq 0.$$

For the region $0 \leq x + ct \leq l, 0 \leq x - ct \leq l$, the solution is given by (18) and (19).

In order to obtain solution for large t, we apply the boundary conditions.

$$\therefore \psi(ct) + \varphi(-ct) = p(t), \qquad t \geq 0, \ldots (25)$$

$$\psi(l + ct) + \varphi(l - ct) = q(t), \qquad t \geq 0. \ \ldots (26)$$

Setting $\alpha = -ct$, we find from (25)

$$\varphi(\alpha) = p\left(-\frac{\alpha}{c}\right) - \psi(-\alpha), \qquad \alpha \leq 0.$$

Setting $\alpha = l + ct$, we find from (25)

$$\psi(\alpha) = q\left(\frac{\alpha - l}{c}\right) - \varphi(2l - \alpha), \qquad \alpha \geq l.$$

Like the case of homogeneous boundary conditions, the expressions of functions ψ and φ can be determined for all $\xi \geq 0$ and for all $\eta \leq 0$ respectively.

$$\varphi(\eta) = p\left(-\frac{\eta}{c}\right) - \psi(-\eta) = p\left(-\frac{\eta}{c}\right) - \frac{1}{2} f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - \frac{D}{2}, 0 \leq -\eta \leq l,$$

$$or - l \leq \eta \leq 0, \ldots (27)$$

$$\psi(\xi) = q\left(\frac{\xi - l}{c}\right) - \varphi(2l - \xi) = q\left(\frac{\xi - l}{c}\right) - \frac{1}{2} f(2l - \xi) + \frac{1}{2c}\int_0^{2l-\xi} g(\tau)d\tau + \frac{D}{2},$$

$$0 \leq 2l - \xi \leq l, or \; l \leq \xi \leq 2l, \ldots (28)$$

$$\varphi(\eta) = p\left(-\frac{\eta}{c}\right) - \psi(-\eta) = p\left(-\frac{\eta}{c}\right) - q\left(\frac{-\eta - l}{c}\right) + \frac{1}{2} f(2l + \eta) - \frac{1}{2c}\int_0^{2l+\eta} g(\tau)d\tau - \frac{D}{2},$$

$$-2l \leq \eta \leq -l \ldots (29)$$

$$\begin{aligned}\psi(\xi) &= q\left(\frac{\xi - l}{c}\right) - \varphi(2l - \xi) \\ &= q\left(\frac{\xi - l}{c}\right) - p\left(\frac{\xi - 2l}{c}\right) + \frac{1}{2} f(\xi - 2l) + \frac{1}{2c}\int_0^{\xi-2l} g(\tau)d\tau + \frac{D}{2},\end{aligned}$$

$$2l \leq \xi \leq 3l, \ldots (30)$$

Problems:

<u>Multiple Choice Questions:</u>

1. Let $u = \psi(x,t)$ be the solution to the initial value problem $u_{tt} = u_{xx}$ for $-\infty < x < \infty, t > 0$ with $u(x,0) = \sin x, u_t(x,0) = \cos x$, then the value of $\psi\left(\frac{\pi}{2}, \frac{\pi}{6}\right)$ is

$$a)\frac{\sqrt{3}}{2}, \qquad b)\frac{1}{2}, \qquad c)\frac{1}{\sqrt{2}}, \quad d)\ 1.$$

Sol. The D'Alembert's solution is

$$u(x,t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau,$$

where $f(x) = \sin x, g(x) = \cos x, c = 1$.

$$\psi\left(\frac{\pi}{2},\frac{\pi}{6}\right)=\frac{1}{2}\left[\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right)+\sin\left(\frac{\pi}{2}-\frac{\pi}{6}\right)\right]+\frac{1}{2c}\int_{\frac{\pi}{2}-\frac{\pi}{6}}^{\frac{\pi}{2}+\frac{\pi}{6}} g(\tau)d\tau=\sin\frac{\pi}{3}+\frac{1}{2}\sin\tau\Bigg|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}=\frac{\sqrt{3}}{2}.$$

Ans (a)

2. The solution of the initial value problem $u_{tt}=4u_{xx},\ t>0,-\infty<x<\infty$, satisfying the conditions $u(x,0)=x, u_t(x,0)=0$ is

$$a)\ x,\quad b)\frac{x^2}{2},\quad c)2x,\quad d)\ 2t$$

The D' Alembert's solution is given by

$$u(x,t)=\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau$$

where $f(x)=x, g(x)=0, c=2.$

$$\therefore u(x,t)=u(x,t)=\frac{1}{2}[(x+2t)+(x-2t)]=x.$$

Ans (a)

....

3. Determine the solution of the initial boundary value problem

$$u_{tt}=4u_{xx},\qquad x>0,t>0,$$

$$u(x,0)=|\sin x|,\qquad x\geq 0,$$

$$u_t(x,0)=0,\qquad x\geq 0,$$

$$u(0,t)=0,\qquad t\geq 0.$$

The PDE corresponds semi-infinite vibrating string with fixed end.

The solution is given by

$$u(x,t)=\begin{cases}\frac{1}{2}[f(x+ct)+f(x-ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, & x>ct\\ \frac{1}{2}[f(x+ct)-f(ct-x)]+\frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)d\tau, & x<ct.\end{cases},$$

where $f(x)=|\sin x|, g(x)=0, c=2.$

$$u(x,t) = \begin{cases} \frac{1}{2}[|\sin(x+2t)| + |\sin(x-2t)|], & x > 2t \\ \frac{1}{2}[|\sin(x+2t)| - |\sin(2t-x)|], & x < 2t. \end{cases}$$

4. Solve the initial value problem described by

$$u_{tt} = c^2 u_{xx}, \qquad x > 0, t > 0,$$

$$u(x,0) = 0, \qquad x \geq 0,$$

$$u_t(x,0) = 0, \qquad x \geq 0,$$

$$u(0,t) = \sin t, \qquad t \geq 0.$$

The PDE corresponds semi-infinite vibrating string with free end (Non-Homogeneous B.C).

The solution is given by

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, & x > ct \\ \frac{1}{2}[f(x+ct) - f(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} g(\tau)d\tau + p\left(t - \frac{x}{c}\right), & x < ct, \end{cases}$$

where $f(x) = 0, g(x) = 0, p(t) = \sin t$.

$$\therefore u(x,t) = \begin{cases} 0, & x > ct \\ \sin\left(t - \frac{x}{c}\right), & x < ct. \end{cases}$$

5. Find the solution of the initial boundary value problem

$$u_{tt} = u_{xx}, \qquad x > 0, t > 0,$$

$$u(x,0) = \cos\left(\frac{\pi x}{2}\right), \qquad x \geq 0,$$

$$u_t(x,0) = 0, \qquad x \geq 0,$$

$$u_x(0,t) = 0, \qquad t \geq 0.$$

The PDE corresponds semi-infinite vibrating string with free end (Homogeneous B.C).

The solution is given by

$$u(x,t) = \begin{cases} \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, & x > ct \\ \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(\tau)d\tau + \int_0^{ct-x} g(\tau)d\tau\right], & x < ct, \end{cases}$$

where $f(x) = \cos\left(\frac{\pi x}{2}\right), g(x) = 0, c = 1.$

$$\therefore u(x,t) = \begin{cases} \frac{1}{2}\left[\cos\left\{\frac{\pi(x+t)}{2}\right\} + \cos\left\{\frac{\pi(x-t)}{2}\right\}\right] = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right), & x > t \\ \frac{1}{2}\left[\cos\left\{\frac{\pi(x+t)}{2}\right\} + \cos\left\{\frac{\pi(t-x)}{2}\right\}\right] = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right), & x < t, \end{cases}$$

$$or, \qquad u(x,t) = \cos\left(\frac{\pi x}{2}\right)\cos\left(\frac{\pi t}{2}\right).$$

6. Determine the solution of the following finite vibrating string problem:

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < l, t > 0,$$

$$u(x,0) = \sin\left(\frac{\pi x}{l}\right), \qquad 0 \le x \le l,$$

$$u_t(x,0) = 0, \qquad 0 \le x \le l,$$

$$u(0,t) = 0 = u(l,t), \qquad t \ge 0.$$

The solution is given by

$$u(x,t) = \psi(x+ct) + \varphi(x-ct)$$

where

$$\psi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \frac{D}{2}, \qquad 0 \le \xi \le l,$$

$$\varphi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{D}{2}, \qquad 0 \le \eta \le l,$$

$$\psi(\xi) = -\frac{1}{2}f(2l-\xi) + \frac{1}{2c}\int_0^{2l-\xi} g(\tau)d\tau + \frac{D}{2}, \qquad l \le \xi \le 2l,$$

$$\varphi(\eta) = -\frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - \frac{D}{2}, \qquad -l \le \eta \le 0,$$

$$\psi(\xi) = \frac{1}{2}f(\xi-2l) + \frac{1}{2c}\int_0^{\xi-2l} g(\tau)d\tau + \frac{D}{2}, \qquad 2l \le \xi \le 3l,$$

$$\varphi(\eta) = \frac{1}{2}f(2l+\eta) - \frac{1}{2c}\int_0^{2l+\eta} g(\tau)d\tau - \frac{D}{2}, -2l \le \eta \le -l,$$

and so on.

Here $f(x) = \sin\left(\frac{\pi x}{l}\right), g(x) = 0.$

The solution is thus

$$\psi(\xi) = \frac{1}{2}\sin\left(\frac{\pi\xi}{l}\right) + \frac{D}{2}, \qquad 0 \le \xi \le l,$$

$$\varphi(\eta) = \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{D}{2}, \qquad 0 \le \eta \le l,$$

$$\psi(\xi) = -\frac{1}{2}\sin\left\{\frac{\pi(2l-\xi)}{l}\right\} + \frac{D}{2} = \frac{1}{2}\sin\left(\frac{\pi\xi}{l}\right) + \frac{D}{2}, \qquad l \le \xi \le 2l,$$

$$\varphi(\eta) = -\frac{1}{2}\sin\left\{\frac{\pi(-\eta)}{l}\right\} = \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{D}{2}, \qquad -l \le \eta \le 0,$$

$$\psi(\xi) = \frac{1}{2}\sin\left\{\frac{\pi(\xi-2l)}{l}\right\} + \frac{D}{2} = \frac{1}{2}\sin\left(\frac{\pi\xi}{l}\right) + \frac{D}{2}, \qquad 2l \le \xi \le 3l,$$

$$\varphi(\eta) = \frac{1}{2}\sin\left\{\frac{\pi(2l+\eta)}{l}\right\} - \frac{D}{2} = \frac{1}{2}\sin\left(\frac{\pi\eta}{l}\right) - \frac{D}{2}, -2l \le \eta \le -l,$$

and so on.

The solution is given by

$$u(x,t) = \psi(x+ct) + \varphi(x-ct) = \frac{1}{2}\left[\sin\frac{\pi(x+ct)}{l} + \sin\frac{\pi(x-ct)}{l}\right] = \sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

for all x s.t $0 \le x \le l, t \ge 0$.

7. Solve the PDE

$$u_{tt} = c^2 u_{xx}, \qquad 0 \le x \le l, t \ge 0,$$

$$u(0,t) = 0 = u(l,t), \qquad t > 0,$$

$$u(x,0) = 0, u_t(x,0) = b\sin^3\frac{\pi x}{l}, \qquad 0 \le x \le l.$$

The solution is given by

$$u(x,t) = \psi(x+ct) + \varphi(x-ct),$$

where

$$\psi(\xi) = \frac{1}{2}f(\xi) + \frac{1}{2c}\int_0^{\xi} g(\tau)d\tau + \frac{D}{2}, \qquad 0 \le \xi \le l,$$

$$\varphi(\eta) = \frac{1}{2}f(\eta) - \frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{D}{2}, \qquad 0 \le \eta \le l,$$

$$\psi(\xi) = -\frac{1}{2}\, f(2l-\xi) + \frac{1}{2c}\int_0^{2l-\xi} g(\tau)d\tau + \frac{D}{2}, \qquad l \le \xi \le 2l,$$

$$\varphi(\eta) = -\frac{1}{2}\, f(-\eta) - \frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - \frac{D}{2}, \qquad -l \le \eta \le 0,$$

$$\psi(\xi) = \frac{1}{2}\, f(\xi-2l) + \frac{1}{2c}\int_0^{\xi-2l} g(\tau)d\tau + \frac{D}{2}, \qquad 2l \le \xi \le 3l,$$

$$\varphi(\eta) = \frac{1}{2}\, f(2l+\eta) - \frac{1}{2c}\int_0^{2l+\eta} g(\tau)d\tau - \frac{D}{2}, -2l \le \eta \le -l,$$

and so on.

Here $f(x) = 0, g(x) = b\sin^3\frac{\pi x}{l}$.

$$\psi(\xi) = \frac{1}{2c}\int_0^{\xi} b\sin^3\frac{\pi\tau}{l}d\tau + \frac{D}{2} = \left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\xi}{l}\right) - 3\cos\left(\frac{\pi\xi}{l}\right) + \frac{8}{3}\right] + \frac{D}{2},$$
$$0 \le \xi \le l,$$

$$\varphi(\eta) = -\frac{1}{2c}\int_0^{\eta} g(\tau)d\tau - \frac{D}{2} = -\left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\eta}{l}\right) - 3\cos\left(\frac{\pi\eta}{l}\right) + \frac{8}{3}\right] - \frac{D}{2},$$
$$0 \le \eta \le l,$$

$$\psi(\xi) = \frac{1}{2c}\int_0^{2l-\xi} g(\tau)d\tau + \frac{D}{2} = \left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left\{\frac{3\pi(2l-\xi)}{l}\right\} - 3\cos\left\{\frac{\pi(2l-\xi)}{l}\right\} + \frac{8}{3}\right] + \frac{D}{2}$$
$$= \left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\xi}{l}\right) - 3\cos\left(\frac{\pi\xi}{l}\right) + \frac{8}{3}\right] + \frac{D}{2}, \qquad l \le \xi \le 2l,$$

$$\varphi(\eta) = -\frac{1}{2c}\int_0^{-\eta} g(\tau)d\tau - \frac{D}{2} = -\left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\eta}{l}\right) - 3\cos\left(\frac{\pi\eta}{l}\right) + \frac{8}{3}\right] - \frac{D}{2},$$
$$-l \le \eta \le 0,$$

$$\psi(\xi) = \frac{1}{2c}\int_0^{\xi-2l} g(\tau)d\tau + \frac{D}{2} = \left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left\{\frac{3\pi(\xi-2l)}{l}\right\} - 3\cos\left\{\frac{\pi(\xi-2l)}{l}\right\} + \frac{8}{3}\right] + \frac{D}{2}$$
$$= \left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\xi}{l}\right) - 3\cos\left(\frac{\pi\xi}{l}\right) + \frac{8}{3}\right] + \frac{D}{2}, \qquad 2l \le \xi \le 3l,$$

$$\varphi(\eta) = -\frac{1}{2c}\int_0^{2l+\eta} g(\tau)d\tau - \frac{D}{2}$$
$$= -\left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left\{\frac{3\pi(\eta+2l)}{l}\right\} - 3\cos\left\{\frac{\pi(\eta+2l)}{l}\right\} + \frac{8}{3}\right] - \frac{D}{2}$$
$$= -\left(\frac{bl}{8\pi c}\right)\left[\frac{1}{3}\cos\left(\frac{3\pi\eta}{l}\right) - 3\cos\left(\frac{\pi\eta}{l}\right) + \frac{8}{3}\right] - \frac{D}{2}, -2l \le \eta \le -l,$$

and so on.

The solution is

$$u(x,t) = \psi(x+ct) + \varphi(x-ct)$$
$$= \frac{bl}{8\pi c}\left[\cos\left\{\frac{3\pi(x+ct)}{l}\right\} - \cos\left\{\frac{3\pi(x-ct)}{l}\right\} - 3\cos\left\{\frac{\pi(x+ct)}{l}\right\} + 3\cos\left\{\frac{\pi(x-ct)}{l}\right\}\right] = \frac{bl}{4\pi c}\left[\sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l} - 3\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l}\right]$$

for all x s.t $0 < x \le l, t \ge 0$.

....

Non-homogeneous wave equation (Infinite string):

Let a string vibrates under an external force $F(x,t)$. The resulting motion is then

$$u_{tt} = c^2u_{xx} + F(x,t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \quad -\infty < x < \infty.$$

Consider the transformation $\qquad y = ct.$

$$\therefore u_t = u_y\frac{dy}{dt} = cu_y, u_{tt} = c^2u_{yy}.$$

The given PDE becomes

$$c^2u_{yy} = c^2u_{xx} + F\left(x,\frac{y}{c}\right), or\ u_{xx} - u_{yy} = -\frac{F\left(x,\frac{y}{c}\right)}{c^2} = F^*(x,y) \ \ ...(1)$$

The initial conditions are

$u(x,0) = f(x),\ u_y(x,0) = \frac{1}{c}g(x) = g^*(x).\ \ ...(2)$

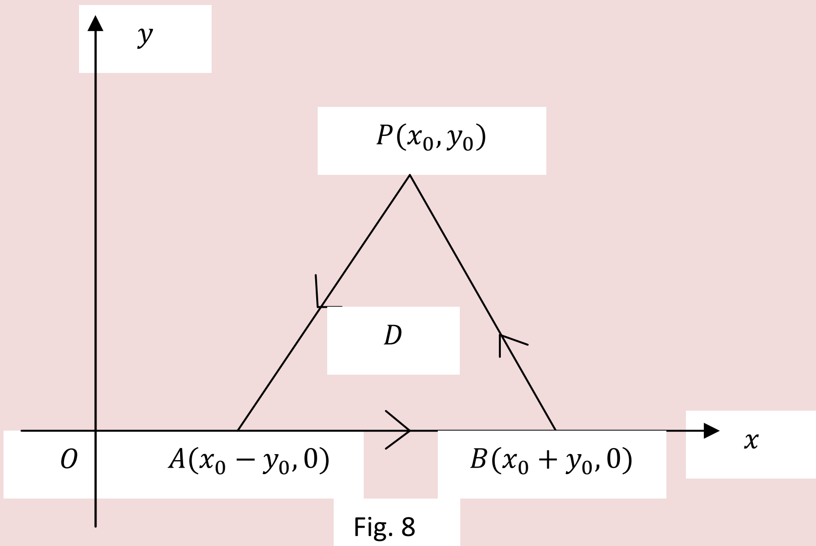


Fig. 8

The characteristic equations $x \pm ct = \text{const}$ of the given PDE becomes $x \pm y = \text{const}$. Let $P(x_0, y_0)$ be a point on the xy −plane. Let PQ be perpendicular to x −axis with foot of the perpendicular $Q(x_0, 0)$. Let PB be the line $x + y = x_0 + y_0$ and PA be the line $x - y = x_0 - y_0$. $\therefore A = (x_0 - y_0, 0), B = (x_0 + y_0, 0)$. Let D be the region bounded by the triangle PAB.

[*Green's Theorem: If D be a closed region in the xy −plane bounded by a simple closed curve C. If M and N are continuous functions of x, y having continuous first order partial derivatives, then* $\int_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$.]

By Green's theorem, we have for the region D bounded by triangle PAB,

$$\iint_D (u_{xx} - u_{yy})dxdy = \int_L (u_y dx + u_x dy),$$

L is the boundary of D.

Now, $\int_L (u_y dx + u_x dy) = \int_{L_0} (u_y dx + u_x dy) + \int_{L_1} (u_y dx + u_x dy) + \int_{L_2} (u_y dx + u_x dy)$.

$$\int_{L_0} (u_y dx + u_x dy) = \int_{x_0-y_0}^{x_0+y_0} (u_y dx + u_x dy) = \int_{x_0-y_0}^{x_0+y_0} u_y dx \ [\because \text{on } x - \text{axis } dy = 0]$$

$$\int_{L_1} (u_y dx + u_x dy) = \int_{L_1} (-u_y dy - u_x dx) \ [\because \text{on } L_1, x + y = x_0 + y_0, \text{or } dx = -dy]$$

$$= -\int_{L_1} (u_x dx + u_y dy) = -\int_{L_1} du = u|_{L_1} = -\{u(x_0, y_0) - u(x_0 + y_0, 0)\}$$
$$= u(x_0 + y_0, 0) - u(x_0, y_0).$$

$$\int_{L_2} (u_y dx + u_x dy) = \int_{L_1} (u_y dy + u_x dx) \ [\because \text{on } L_1, x - y = x_0 - y_0, \text{or } dx = dy]$$

$$= \int_{L_2} du = u|_{L_2} = u(x_0 - y_0, 0) - u(x_0, y_0).$$

$$\int_L (u_y dx + u_x dy) = \int_{x_0-y_0}^{x_0+y_0} u_y dx + u(x_0 + y_0, 0) - u(x_0, y_0) + u(x_0 - y_0, 0) - u(x_0, y_0)$$

$$= u(x_0+y_0,0)+u(x_0-y_0,0)-2u(x_0,y_0)+\int_{x_0-y_0}^{x_0+y_0} u_y\,dx.$$

$$\iint_D (u_{xx}-u_{yy})dxdy = f(x_0+y_0)+f(x_0-y_0)-2u(x_0,y_0)+\int_{x_0-y_0}^{x_0+y_0} g^*(x)dx\,,$$

$$or, \iint_D F^*(x,y)dxdy = f(x_0+y_0)+f(x_0-y_0)-2u(x_0,y_0)+\int_{x_0-y_0}^{x_0+y_0} g^*(x)dx\,,$$

$$or, u(x_0,y_0)=\frac{1}{2}[f(x_0+y_0)+f(x_0-y_0)]+\frac{1}{2}\int_{x_0-y_0}^{x_0+y_0} g^*(x)dx$$

$$-\frac{1}{2}\int_{y=0}^{y_0}\int_{x=y+x_0-y_0}^{x=-y+x_0+y_0} F^*(x,y)dxdy \quad \ldots(3)$$

Since x_0, y_0 are chosen arbitrarily, replace x_0 by x and y_0 by y in (3), we obtain $u(x,y)$.

(3) also takes the form

$$u(x_0,t_0)=\frac{1}{2}[f(x_0+ct_0)+f(x_0-ct_0)]+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau$$

$$-\frac{1}{2}\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0}\left(-\frac{F(x,t)}{c^2}\right)dx(cdt)$$

$$=\frac{1}{2}[f(x_0+ct_0)+f(x_0-ct_0)]+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau$$

$$+\frac{1}{2c}\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} F(x,t)dxdt. \quad \ldots(4)$$

Replacing x_0 by x and t_0 by t in (4), we obtain $u(x,t)$.

Problem:

1. Solve the initial value problem

$$u_{tt}-c^2u_{xx}=e^x, \quad x\in R,$$

$$u(x,0)=5, \qquad u_t(x,0)=x^2, \qquad x\in R.$$

$$u(x_0,t_0)=\frac{1}{2}[f(x_0+ct_0)+f(x_0-ct_0)]+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau$$

$$+\frac{1}{2c}\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} F(x,t)dxdt.$$

Now, $$\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} F(x,t)dxdt=\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} e^x dxdt$$

$$=\int_{t=0}^{t_0}[e^{-ct+x_0+ct_0}-e^{ct+x_0-ct_0}]dt=\frac{e^{x_0}}{c}[-2+e^{ct_0}+e^{-ct_0}]$$

$$=\frac{4e^{x_0}}{c}\sinh^2\frac{ct_0}{2}.$$

$$\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau=\int_{x_0-ct_0}^{x_0+ct_0}\tau^2 d\tau=\frac{1}{3}[(x_0+ct_0)^3-(x_0-ct_0)^3]=\frac{2t_0}{3}[7c^2t_0^2-3x_0^2].$$

$$u(x_0,t_0)=\frac{1}{2}[5+5]+\frac{t_0}{3}(7c^2t_0^2-3x_0^2)+\frac{2e^{x_0}}{c^2}\sinh^2\frac{ct_0}{2}.$$

Since x_0, t_0 are arbitrary point in the $xt-$ plane, replacing x_0 by x and t_0 by t, we find

$$u(x,t)=5+\frac{t}{3}(7c^2t^2-3x^2)+\frac{2e^x}{c^2}\sinh^2\frac{ct}{2}.$$

2. Solve the IVP:

$$u_{tt}-c^2u_{xx}=xe^t, \qquad x\in R,$$

$$u(x,0)=\sin x\,, u_t(x,0)=0, \qquad x\in R\,.$$

$$u(x_0,t_0)=\frac{1}{2}[f(x_0+ct_0)+f(x_0-ct_0)]+\frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(\tau)d\tau$$

$$+\frac{1}{2c}\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} F(x,t)dxdt.$$

$$u(x_0,t_0)=\frac{1}{2}[\sin(x_0+ct_0)+\sin(x_0-ct_0)]+\frac{1}{2c}\int_{t=0}^{t_0}\int_{x=ct+x_0-ct_0}^{x=-ct+x_0+ct_0} xe^t dxdt$$

$$= \sin x_0 \cos(ct_0) + \frac{1}{2c}\left[\int_{t=0}^{t_0} e^t\{(-ct + x_0 + ct_0)^2 - (ct + x_0 - ct_0)^2\}\,dt\right]$$

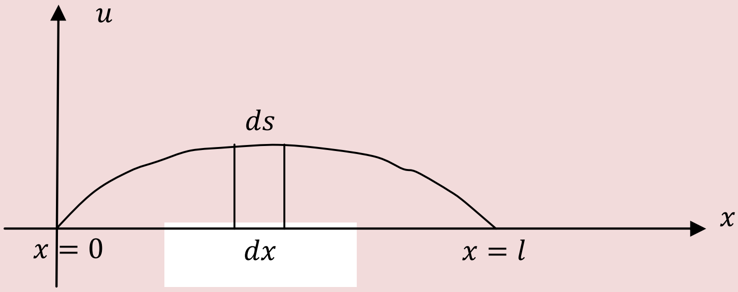
$$= \sin x_0 \cos(ct_0) + \frac{1}{4c}\int_{t=0}^{t_0} e^t\{-4cx_0t + 4cx_0t_0\}dt = \sin x_0 \cos(ct_0) + x_0(e^{t_0} - t_0 - 1)$$

$$u(x,t) = \sin x \cos(ct) + x(e^t - t - 1).$$

....

Total Energy of a String fixed at the points $x = 0$ and $x = l$:

Let the string be under small transverse vibration. Let T be the tension of the uniform string of density ρ.



The K.E. of an element dx of the string is given by $\frac{1}{2}\rho dx\left(\frac{\partial u}{\partial t}\right)^2$.

$$\text{Total K.E.} = \frac{1}{2}\rho\int_0^l \left(\frac{\partial u}{\partial t}\right)^2 dx = \frac{T}{2}\int_0^l \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 dx \left[\because c^2 = \frac{T}{\rho}\right].$$

Now, $ds = \sqrt{(dx)^2 + (du)^2}$, or, $\frac{ds}{dx} = \sqrt{1 + \left(\frac{du}{dx}\right)^2} = 1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \cdots \cong 1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2$.

$$ds \cong \left[1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2\right]dx.$$

Due to the motion the stretch in element dx of the string is

$$ds - dx \cong \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Hence, work done by this element dx is given by

$$\frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

$$\text{Total workdone} = \text{Total P.E.} = \int_0^l \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx.$$

$$\text{K.E.} + \text{P.E.} = \frac{T}{2}\int_0^l \frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 dx + \int_0^l \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx = \frac{T}{2}\left[\int_0^l \left\{\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\} dx\right].$$

Linear Operator:

An operator L is said to be linear if it satisfies

$$L[\alpha u + \beta v] = \alpha L[u] + \beta L[v], \qquad \alpha \text{ and } \beta \text{ are scalars.}$$

E.g.,

Laplace's equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ can be expressed as

$$L[u] = 0 \text{ where } L \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ can be expressed as

$$M[u] = 0 \text{ where } M \equiv \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}.$$

Hear equation: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ can be expressed as

$$N[u] = 0 \text{ where } N \equiv \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2}.$$

L, M, N are all linear operators.

Superposition Principle:

Let us consider the following initial value problem

$$\left.\begin{aligned} &u_{tt} = c^2 u_{xx} + H(x,t), \qquad 0 < x < l, t > 0, \\ &u(x,0) = g_1(x), \qquad u_t(x,0) = g_2(x), \qquad 0 \le x \le l, \\ &u(0,t) = g_3(t), \qquad u(l,t) = g_4(t), \qquad t \ge 0. \end{aligned}\right\} \quad ..(A)$$

(A) canbe written in the operator form as

$$L[u] = H,$$

$$M_1[u] = g_1,$$

$$M_2[u] = g_2,$$

$$M_3[u] = g_3$$

$$M_4[u] = g_4,$$

where

$$L \equiv \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}, M_1[u] = u(x,0), M_2[u] = u_t(x,0), M_3[u] = u(0,t), M_4[u] = u(l,t).$$

Let us now consider a more generalized problem (B)

$$L[u] = H,$$

$$\left.\begin{aligned} M_1[u] &= g_1, \\ M_2[u] &= g_2, \\ &\dots \\ M_{n-1}[u] &= g_{n-1}. \end{aligned}\right\} \quad ..(B)$$

Due to linearity of the operators (B) can be rewritten as combination of the following subproblems:

$$L[u_1] = H,$$

$$\left.\begin{aligned} M_1[u_1] &= 0, \\ M_2[u_1] &= 0, \\ &\dots \\ M_{n-1}[u_1] &= 0, \end{aligned}\right\} \quad ..(C)$$

$$L[u_2] = 0,$$

$$\left.\begin{aligned} M_1[u_2] &= g_1, \\ M_2[u_2] &= 0, \\ &\dots \\ M_{n-1}[u_2] &= 0, \end{aligned}\right\} \quad ..(D)$$

$$\dots$$

$$L[u_n] = 0,$$

$$\left.\begin{aligned} M_1[u_n] &= 0, \\ M_2[u_n] &= 0, \\ &\dots \\ M_{n-1}[u_n] &= g_{n-1}. \end{aligned}\right\} \quad ..(E)$$

Then solution of (B) is given by

$$u = \sum_{i=1}^{n} u_i \quad \dots(F)$$

Let us now consider one of the subproblem, say, (C). Suppose there exists a sequence of functions $\varphi_1, \varphi_2, ..,$ (finite or infinite) which satisfy the homogeneous system

$$\left.\begin{aligned} L[\varphi_i] &= 0, \\ M_2[\varphi_i] &= 0, \\ &\dots \\ M_{n-1}[\varphi_i] &= 0. \end{aligned}\right\} \quad ..(G)$$

and g_1 is expressed as

$$g_1 = c_1 M_1[\varphi_1] + c_2 M_1[\varphi_2] + \cdots + c_n M_1[\varphi_n] + \cdots \quad (H)$$

Then the linear combination

$$u_2 = c_1\varphi_1 + c_2\varphi_2 + \cdots + c_n\varphi_n + \cdots \quad (I)$$

is the solution of (D).

If there be infinite no. of terms in the expression of u_2 in (I), then the series need to be convergent and sufficiently differentiable.

Separation of Variables for solving initial boundary value problem:

Consider the second-order linear homogeneous PDE

$$a'u_{x'x'} + b'u_{x'y'} + c'u_{y'y'} + d'u'_x + e'u'_y + f'u = 0 \quad \dots(1)$$

where a', b', c', d', e', f' are all functions of x' and y'.

We transform the equ. (1) into canonical form by the following transformation

$$x = x(x', y'), \quad y = y(x', y') \quad \dots(2)$$

so that $\frac{\delta(x,y)}{\delta(x',y')} = \begin{vmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{vmatrix} \neq 0.$

Let under the transformation(2), (1) is transformed into the canonical form

$$au_{xx} + cu_{yy} + du_x + eu_y + fu = 0 \quad \dots (3)$$

where the coefficients a, c, d, e, f all are functions of x, y.

(3) is hyperbolic if $a + c = 0$, parabolic if $a = 0$ or $c = 0$, hyperbolic if $a = c$.

Let us suppose (3) has a solution of the form

$$u(x, y) = X(x)Y(y) \quad \dots (4)$$

$$u_x = X'Y, u_{xx} = X''Y, u_y = XY', u_{yy} = XY'' \quad \dots (5)$$

With the expressions of (4) and (5), the PDE (3) takes the form

$$aX''Y + cXY'' + dX'Y + eXY' + fXY = 0. \quad \dots (6)$$

Let us further suppose that the following arrangement of (6) can be made.

$$\left[a_1(x)\left(\frac{X''}{X}\right) + a_2(x)\left(\frac{X'}{X}\right) + a_3(x)\right] = -\left[b_1(y)\left(\frac{Y''}{Y}\right) + b_2(y)\left(\frac{Y'}{Y}\right) + b_3(y)\right]. \quad \dots (7)$$

The left side of (7) is a function of x only and the right side of (7) is a function of y only.

On differentiation of (7) w.r.t x,

$$\frac{d}{dx}\left[a_1(x)\left(\frac{X''}{X}\right) + a_2(x)\left(\frac{X'}{X}\right) + a_3(x)\right] = 0. \quad \dots (8)$$

On integration of (8) w.r.t. x,

$$a_1(x)\left(\frac{X''}{X}\right) + a_2(x)\left(\frac{X'}{X}\right) + a_3(x) = k, \qquad \dots (9)$$

k being the separation constant.

From (7) and (9), we find

$$b_1(y)\left(\frac{Y''}{Y}\right) + b_2(y)\left(\frac{Y'}{Y}\right) + b_3(y) = -k. \quad \dots (10)$$

(9) and (10) gives

$$\left.\begin{aligned} a_1(x)X'' + a_2(x)X' + (a_3(x) - k)X = 0, \\ b_1(y)Y'' + b_2(y)Y' + (b_3(y) - k)Y = 0. \end{aligned}\right\} \quad \dots(11)$$

Solving the ODE in (11) we obtain $X(x)$ and $Y(y)$. Hence $u(x, y)$ is obtained.

Finite Vibrating String:

<u>Homogeneous wave equation with homogeneous boundary conditions:</u>

Let a homogeneous string of tension T be stretched along the x −axis at $x = 0$ and $x = l$. Let the string be under transverse vibration. If no external force is present, the problem is given by

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \qquad t > 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \qquad 0 \leq x \leq l,$$

$$u(0, t) = 0 = u(l, t), \qquad t > 0.$$

f and g are respectively the initial displacement and initial velocity.

Let $u(x, t)$ be expressed as

$$u(x, t) = X(x)T(t). \qquad \dots(12)$$

Substituting (12) into the given PDE, we find

$$X(x)T''(t) = c^2 X''(x)T(t),$$

$$or, \frac{X''}{X} = \frac{T''}{c^2 T} = k, \qquad \dots(13)$$

k is the separation constant.

Now three cases may arise.

Case-I: Let $k > 0$. Take $k = \lambda^2$. This produces

$$X'' = \lambda^2 X, or\ X(x) = Ae^{\lambda x} + Be^{-\lambda x},$$

$$T'' = c^2\lambda^2 T, or\ T(t) = Ce^{\lambda ct} + De^{-\lambda ct}.$$

$$\therefore u(x, t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda ct} + De^{-\lambda ct}\right).$$

The B.Cs $u(0, t) = 0 = u(l, t)$ give

$$\left.A + B = 0,\right\} \quad \dots(14)$$

$$Ae^{lt} + Be^{-lt} = 0.$$

The coefficient determinant of system (14) is $\begin{vmatrix} 1 & 1 \\ e^{lt} & e^{-lt} \end{vmatrix} \neq 0$. Hence, the system (14) possesses only trivial solution $A = B = 0$. Hence, only trivial solution is found in this case.

Case-II: Let $k = 0$.

$$X'' = 0, or\ X(x) = A + Bx,$$

$$T''(t) = 0, or\ T(t) = C + Dt.$$

$$\therefore u(x,t) = (A + Bx)(C + Dt).$$

On application of B.Cs $u(0,t) = 0 = u(l,t)$,

$$A = 0 = B.$$

Hence, only trivial solution is found in this case also.

Case-III: Let $k < 0$. Take $k = -\lambda^2$.

$$\therefore X'' = -\lambda^2 X, or\ X(x) = A\cos\lambda x + B\sin\lambda x,$$

$$T''(t) = -\lambda^2 c^2 T, or\ T(t) = C\cos\lambda ct + D\sin\lambda ct.$$

Now the B.Cs produce

$$A = 0,$$

$$A\cos\lambda l + B\sin\lambda l = 0, or\ B\sin\lambda l = 0, or\ \sin\lambda l = 0,$$

$$or\ \lambda_n = \frac{n\pi}{l}, n \in N.$$

With $\lambda = \lambda_n = \frac{n\pi}{l}, X_n(x) = B_n \sin\lambda_n x = B_n \sin\left(\frac{n\pi x}{l}\right)$.

[$\lambda_n{}'s$ are called eigen value and $X_n{}'s$ are called eigen value.]

$$\therefore u_n(x,t) = B_n \sin\left(\frac{n\pi x}{l}\right)\left\{C_n \cos\left(\frac{nc\pi t}{l}\right) + D_n \sin\left(\frac{nc\pi t}{l}\right)\right\}$$

$$= \sin\left(\frac{n\pi x}{l}\right)\left\{a_n \cos\left(\frac{nc\pi t}{l}\right) + b_n \sin\left(\frac{nc\pi t}{l}\right)\right\},$$

where $a_n(= B_n C_n), b_n(= B_n D_n)$ are arbitrary constants.

Since, the PDE is linear and homogeneous, by superposition principle, the solution is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\left\{a_n \cos\left(\frac{nc\pi t}{l}\right) + b_n \sin\left(\frac{nc\pi t}{l}\right)\right\}. \quad \dots(15)$$

Now, apply the I.Cs $u(x,0)=f(x),\quad u_t(x,0)=g(x)$.

$$\therefore f(x)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{and } g(x)=\sum_{n=1}^{\infty} b_n\left(\frac{nc\pi}{l}\right)\sin\left(\frac{n\pi x}{l}\right).$$

The two above series are half range Fourier series.

$$\therefore a_n=\frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx,$$

$$b_n=\frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx. \quad \dots(16)$$

The functions $u_n(x,t)=\sin\left(\frac{n\pi x}{l}\right)\left\{a_n\cos\left(\frac{nc\pi t}{l}\right)+b_n\sin\left(\frac{nc\pi t}{l}\right)\right\}$ are called normal modes of vibration. $\omega_n=\frac{n\pi c}{l}$ are called circular frequencies, or $\nu_n=\frac{\omega_n}{2\pi}=\frac{nc}{2l}$ are called angular frequencies.

Note: If the initial velocity $u_t(x,0)=0$, then

$$u_n(x,t)=a_n\cos\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right),\qquad n=1,2,3,\dots,$$

$$or, u_n(x,t)=\frac{a_n}{2}\left[\sin\frac{n\pi}{l}(x-ct)+\sin\frac{n\pi}{l}(x+ct)\right]. \quad \dots(17)$$

(17) is a stationary wave profile. At any instant t, (17) represents a sine curve of amplitude $a_n\cos\left(\frac{nc\pi t}{l}\right)$. Thus, a wave profile of this nature does not propagate. The curve intersects the x −axis at $x=0,\frac{l}{n},\frac{2l}{n},\dots$ These points are called nodes, and the intermediate points where the amplitude is maximum are called antinodes.

<u>Non-homogeneous wave equation with non-homogeneous boundary conditions;</u>

Consider the initial-boundary value problem

$$u_{tt}=c^2u_{xx}+F(x),\quad 0<x<l,\qquad t>0,$$

$$u(x,0)=f(x),\qquad u_t(x,0)=g(x),\qquad 0\le x\le l,$$

$$u(0,t)=A,\qquad u(l,t)=B,\qquad t>0.\ [\text{Non}-\text{homogeneous B.Cs}]$$

Let us assume a solution of the form

$$u(x,t) = v(x,t) + W(x). \quad \ldots(18)$$

Then the PDE becomes

$$v_{tt} = c^2 v_{xx} + c^2 W''(x) + F(x), \quad 0 < x < l,$$

$$v(x,0) + W(x) = f(x), \quad 0 < x < l,$$

$$v_t(x,0) = g(x), \quad 0 < x < l,$$

$$v(0,t) + W(0) = A, \qquad v(l,t) + W(l) = B.$$

Let us now suppose $W(x)$ to be the solution of the ODE

$$c^2 W''(x) + F(x) = 0, \qquad W(0) = A, \qquad W(l) = B. \quad \ldots(19)$$

Then $v(x,t)$ satisfies the initial-boundary value problem

$$\left.\begin{aligned} &v_{tt} - c^2 v_{xx} = 0, \ 0 < x < l, \ t > 0, \\ &v(x,0) = f(x) - W(x) = f'(x), \ 0 \le x \le l, \\ &v_t(x,0) = g(x), \ 0 < x < l, \\ &v(0,t) = v(l,t) = 0, \qquad t \ge 0. \end{aligned}\right\} \quad \ldots(20)$$

Solution of (20) is given by (15) and (16).

For the ODE in (19),

$$W''(x) = -\frac{1}{c^2} F(x).$$

On integration,

$$W'(x) = -\int_0^x \frac{1}{c^2} F(\xi) d\xi + K.$$

Again on integration,

$$W(x) = -\int_0^x \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta + Kx + K', \qquad K \text{ and } K' \text{are integration constants.}$$

On application of the B.Cs, $W(0) = A, W(l) = B,$

$$K' = A, B = -\int_0^l \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta + Kl + A, or\ K = \frac{B-A}{l} + \frac{1}{l} \int_0^l \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta.$$

$$W(x) = -\int_0^x \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta + \left[\frac{B-A}{l} + \frac{1}{l} \int_0^l \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta \right] x + A,$$

$$or, W(x) = A + \left(\frac{B-A}{l} \right) x + \frac{x}{l} \int_0^l \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta - \int_0^x \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta \ldots (21)$$

After finding $W(x)$ by (21), $u(x,t)$ is given by

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left\{ a_n \cos\left(\frac{nc\pi t}{l}\right) + b_n \sin\left(\frac{nc\pi t}{l}\right) \right\} + A + \left(\frac{B-A}{l}\right) x$$
$$+ \frac{x}{l} \int_0^l \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta - \int_0^x \left\{ \int_0^\eta \frac{1}{c^2} F(\xi) d\xi \right\} d\eta, \ldots (22)$$

where a_n, b_n are given by

$$a_n = \frac{2}{l} \int_0^l f'(x) \sin\left(\frac{n\pi x}{l}\right) dx,$$

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx. \ \ldots (23)$$

Uniqueness Theorem:

There exists unique solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

satisfying the initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \qquad 0 \le x \le l,$$

and the boundary conditions

$$u(0,t) = 0, \qquad u(l,t) = 0, \qquad t \ge 0,$$

where $u(x,t)$ is a twice continuously differentiable function of x and t.

Proof: Let u_1, u_2 be two solutions of the wave equation. Hence, u_1 and u_2 satisfy the given initial boundary value problem.

$$\text{Let } v = u_1 - u_2.$$

$\therefore v$ satisfies

$$v_{tt} = c^2 v_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$v(x,0) = 0, \qquad v_t(x,0) = 0, \qquad 0 \le x \le l,$$

$$v(0,t) = 0, \qquad v(l,t) = 0, \qquad t \ge 0.$$

Our objective is to prove $v(x,t)$ to be identically zero. To do this, consider the total energy integral

$$E(t) = \frac{T}{2}\left[\int_0^l \left\{\frac{1}{c^2}\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right\} dx\right] = \frac{T}{2}\int_0^l \left\{\frac{1}{c^2} v_t^2 + v_x^2\right\} dx.$$

Since u_1, u_2 are twice continuously differentiable functions of x and t, v is twice continuously differentiable function of x, t.

Now,

$$\frac{dE}{dt} = \frac{T}{2} \times 2 \int_0^l \left\{\frac{1}{c^2} v_t v_{tt} + v_x v_{xt}\right\} dx = T\left[\int_0^l \frac{1}{c^2} v_t v_{tt} dx + \left\{ v_x v_t\Big|_0^l - \int_0^l v_{xx} v_t dx \right\}\right]$$

$$[\because v(l,t) = 0, v(0,t) = 0, \therefore v_t(l,t) = 0, v_t(0,t) = 0]$$

$$= T\int_0^l \left\{\frac{1}{c^2} v_t v_{tt} - v_{xx} v_t\right\} dx = \frac{T}{c^2}\int_0^l \{v_{tt} - c^2 v_{xx}\} v_t dx = 0$$

$$\therefore \frac{dE}{dt} = 0, or\ E(t) = \text{constant} = c \text{ (say)}.$$

Now,

$$E(0) = \frac{T}{2}\int_0^l \left\{\frac{1}{c^2} v_t^2(x,0) + v_x^2(x,0)\right\} dx.$$

$$v(x,0) = 0, \therefore v_x(x,0) = 0. \text{ Also, } v_t(x,0) = 0.$$

$$\therefore E(0) = 0, or\ c = 0.$$

$$\therefore E(t) = 0.$$

$$\text{Now, } E(t) = \frac{T}{2}\int_0^l \left\{\frac{1}{c^2}v_t^2 + v_x^2\right\} dx = 0 \text{ which is possible if and only if } v_x = 0 = v_t.$$

$$v_x = 0 \text{ gives } v(x,t) = h(t). \text{Again, } v(0,t) = 0. \therefore h(t) = 0, or\ v(x,t) = 0.$$

$$\therefore u_1(x,t) = u_2(x,t).$$

Non-homogeneous wave equation (Finite String):

<u>Homogeneous boundary conditions:</u>

Let us take the following initial-boundary value problem

$$u_{tt} = c^2 u_{xx} + h(x,t), \quad 0 < x < l, \ \ t > 0,$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \ \ 0 \le x \le l,$$

$$u(0,t) = 0 = \ u(l,t), \qquad t > 0. \qquad (\text{Homogeneous B. Cs})$$

Let us suppose that $u(x,t)$ is of the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi x}{l}\right) \quad \dots (24)$$

where $u_n(t)$ are to be determined. Note that $u(x,t)$ as defined in (24) satisfies the B.Cs $u(0,t) = u(l,t) = 0$.

Let $h(x,t)$ be of the form

$$h(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{l}\right). \quad \dots (25)$$

Thus,

$$h_n(t) = \frac{2}{l}\int_0^l h(x,t) \sin\left(\frac{n\pi x}{l}\right) dx. \quad \dots (26)$$

Assuming the series (24) to be uniformly convergent, we find u_{tt}, u_{xx} from (24) and substitute into the given PDE. This produces

$$\sum_{n=1}^{\infty} u_n''(t) \sin\left(\frac{n\pi x}{l}\right) = -c^2 \sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 u_n(t) \sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} h_n(t) \sin\left(\frac{n\pi x}{l}\right),$$

$$or, \sum_{n=1}^{\infty}\{u_n''(t)+\lambda_n^2 u_n(t)\}\sin\left(\frac{n\pi x}{l}\right)=\sum_{n=1}^{\infty} h_n(t)\sin\left(\frac{n\pi x}{l}\right) \quad \dots (27)$$

where $\lambda_n = \left(\frac{n\pi c}{l}\right)$.

Multiplying (27) by $\sin\left(\frac{m\pi x}{l}\right)$ and integrating over $x=0$ to $x=l$, we obtain

$$u_m''(t)+\lambda_m^2 u_m(t)=h_m(t). \quad \dots (28)$$

$$\left[\because \int_0^l \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx=\begin{cases}\frac{l}{2}, & n=m,\\ 0 & n\neq m.\end{cases}\right]$$

Our task is now to obtain solution of ODE (28).

Let $u_m^c(t)$ be the solution of the corresponding homogeneous differential equation of (28).

$$\therefore u_m^c(t)=A_m\cos\lambda_m t+B_m\sin\lambda_m t.$$

For particular integral $u_m^p(t)$, let us suppose

$$u_m^p(t)=A(t)\cos\lambda_m t+B(t)\sin\lambda_m t.$$

$$u'^p_m(t)=-\lambda_m A(t)\sin\lambda_m t+\lambda_m B(t)\cos\lambda_m t+A'(t)\cos\lambda_m t+B'(t)\sin\lambda_m t.$$

Choose $A(t), B(t)$ such that

$$A'(t)\cos\lambda_m t+B'(t)\sin\lambda_m t=0 \quad \dots (29)$$

$$u''^p_m(t)=-\lambda_m^2 A(t)\sin\lambda_m t-\lambda_m^2 B(t)\cos\lambda_m t-\lambda_m A'(t)\sin\lambda_m t+\lambda_m B'(t)\cos\lambda_m t.$$

Substituting $u_m^p(t), u'^p_m(t), u'^p_m(t)$ into (28), we find

$$A'(t)\sin\lambda_m t+B'(t)\cos\lambda_m t=\frac{1}{\lambda_m}h_m(t). \quad \dots (30)$$

Solving (29) and (30),

$$A'(t)=-\frac{h_m(t)\sin\lambda_m t}{\lambda_m}, \qquad B'^{(t)}=-\frac{h_m(t)\cos\lambda_m t}{\lambda_m}.$$

$$\therefore A(t)=\int_0^t -\frac{h_m(\tau)\sin\lambda_m\tau}{\lambda_m}\,d\tau, \qquad B(t)=\int_0^t -\frac{h_m(\tau)\cos\lambda_m\tau}{\lambda_m}\,d\tau.$$

$$\therefore u_m^p(t)=\int_0^t \frac{h_m(\tau)\sin\lambda_m(t-\tau)}{\lambda_m}\,d\tau.$$

$$\therefore u_m(t) = u_m^c(t) + u_m^p(t)$$

$$= A_m \cos\lambda_m t + B_m \sin\lambda_m t + \int_0^t \frac{h_m(\tau)\sin\lambda_m(t-\tau)}{\lambda_m}\,d\tau. \quad \dots(31)$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} u_n(t)$$

$$= \sum_{n=1}^{\infty}\left\{A_n \cos\lambda_n t + B_n \sin\lambda_n t + \int_0^t \frac{h_n(\tau)\sin\lambda_n(t-\tau)}{\lambda_n}\,d\tau\right\}\sin\left(\frac{n\pi x}{l}\right). \dots(32)$$

Applying the initial conditions

$$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right), \quad g(x) = \sum_{n=1}^{\infty} \lambda_n A_n \sin\left(\frac{n\pi x}{l}\right).$$

$$\therefore A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad B_n = \frac{2}{\lambda_n l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx. \dots(33)$$

<u>Non-homogeneous boundary conditions:</u>

Consider the problem

$$u_{tt} = c^2 u_{xx} + h(x,t), \quad 0 < x < l, \;\; t > 0,$$

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \;\; 0 \le x \le l,$$

$$u(0,t) = p(t), \qquad u(l,t) = q(t), \qquad t > 0. \qquad (\text{Non} - \text{homogeneous B.Cs})$$

Let the solution of $u(x,t)$ be of the form

$$u(x,t) = v(x,t) + w(x,t). \quad \dots(34)$$

Substituting (34) into the PDE,

$$v_{tt} - c^2 v_{xx} = h(x,t) - w_{tt} + c^2 w_{xx}, \quad 0 < x < l, \;\; t > 0,$$

$$v(x,0) = f(x) - w(x,0), \qquad v_t(x,0) = g(x) - w_t(x,0), \;\; 0 \le x \le l,$$

$$v(0,t) = p(t) - w(0,t), \qquad v(l,t) = q(t) - w(l,t), \qquad t > 0.$$

Let us assume $w(x,t)$ satisfy the PDE

$$w_{xx} = 0, \ 0 < x < l, \quad \ldots (35)$$
$$w(0,t) = p(t), \qquad w(l,t) = q(t), \qquad t > 0.$$

Integrating $w_{xx} = 0$ w.r.t x, we find

$$w_x(x,t) = r(t).$$

Again, integrating w.r.t x,

$$w(x,t) = r(t)x + s(t).$$

Applying the B.Cs , $s(t) = p(t), r(t) = \frac{q(t)-p(t)}{l}$.

$$\therefore w(x,t) = \frac{q(t) - p(t)}{l} x + p(t). \ldots (36)$$

$v(x,t)$ satisfies

$$\left.\begin{aligned} v_{tt} - c^2 v_{xx} = h(x,t) - w_{tt} = \bar{h}(x,t), \quad 0 < x < l, \ \ t > 0, \\ v(x,0) = f(x) - w(x,0) = f'(x), \ \ 0 \le x \le l \\ v_t(x,0) = g(x) - w_t(x,0) = g'(x), \ \ 0 \le x \le l, \\ v(0,t) = 0, \qquad v(l,t) = 0, \qquad t > 0. \ \ \text{(Homogeneous B.Cs)} \end{aligned}\right\} \ldots (37)$$

Assuming $\bar{h}(x,t) = \sum_{n=1}^{\infty} \bar{h}_n(t) \sin\left(\frac{n\pi x}{l}\right)$, solution of (37) is given by

$$v(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \cos \lambda_n t + B_n \sin \lambda_n t + \int_0^t \frac{\bar{h}_n(\tau) \sin \lambda_n (t - \tau)}{\lambda_n} d\tau \right\} \sin\left(\frac{n\pi x}{l}\right) \ldots (38)$$

with

$$A_n = \frac{2}{l} \int_0^l f'(x) \sin\left(\frac{n\pi x}{l}\right) dx, \ \ B_n = \frac{2}{\lambda_n l} \int_0^l g'(x) \sin\left(\frac{n\pi x}{l}\right) dx, \qquad \lambda_n = \left(\frac{n\pi c}{l}\right). \ \ldots (39)$$

Hence,

$$\begin{aligned} u(x,t) = {} & \frac{q(t) - p(t)}{l} x + p(t) \\ & + \sum_{n=1}^{\infty} \left\{ A_n \cos \lambda_n t + B_n \sin \lambda_n t \right. \\ & \left. + \int_0^t \frac{\bar{h}_n(\tau) \sin \lambda_n (t - \tau)}{\lambda_n} d\tau \right\} \sin\left(\frac{n\pi x}{l}\right). \ \ldots (40) \end{aligned}$$

Problems:

Determine the solution of the following initial boundary value problems:

1.

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \pi, \qquad t > 0,$$

$$u(x,0) = 0, \;\; u_t(x,0) = 8\sin^2 x, \quad 0 \le x \le \pi,$$

$$u(0,t) = 0 = u(\pi,t), \quad t > 0.$$

2.

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \pi, \qquad t > 0,$$

$$u(x,0) = \sin x, \;\; u_t(x,0) = x^2 - \pi x, \quad 0 \le x \le \pi,$$

$$u(0,t) = 0 = u(\pi,t), \quad t > 0.$$

3.

$$u_{tt} = c^2 u_{xx}, \;\; 0 < x < \pi, \qquad t > 0,$$

$$u(x,0) = \cos x, \;\; u_t(x,0) = 0, \quad 0 \le x \le \pi,$$

$$u_x(0,t) = 0 = u_x(\pi,t), \quad t > 0.$$

4.

$$u_{tt} = c^2 u_{xx} + Ax, \;\; 0 < x < 1, \qquad t > 0,$$

$$u(x,0) = 0, \;\; u_t(x,0) = 0, \quad 0 \le x \le 1,$$

$$u(0,t) = 0 = u(1,t), \quad t > 0.$$

5.

$$u_{tt} = c^2 u_{xx} + x^2, \;\; 0 < x < 1, \qquad t > 0,$$

$$u(x,0) = x, \;\; u_t(x,0) = 0, \quad 0 \le x \le 1,$$

$$u(0,t) = 0, \qquad u(\pi,t) = 1, \quad t > 0.$$

……

Heat- Conduction Problem:

<u>Homogeneous boundary conditions</u>

Let us first consider a homogeneous rod of length l. Let us further assume the rod to be thin enough that the heat is distributed equally over the cross –section at time t. The surface of the rod is insulated so that there is no heat loss through the boundary. The temperature of the rod is then governed by the following initial boundary value problem:

$$u_t = ku_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$u(0,t) = 0 = u(l,t), \qquad t > 0, \quad [\text{Homogeneous B.Cs}]$$

$$u(x,0) = f(x), \quad 0 \le x \le l.$$

Let us separate the variables x, t of u as

$$u(x,t) = X(x)T(t).$$

$$\therefore u_t = X(x)T'(t), \qquad u_x = X'(x)T(t), \qquad u_{xx} = X''(x)T(t).$$

Substituting the expressions of u_t, u_{xx} into the PDE, we find

$$kX''(x)T(t) = X(x)T('t),$$

$$or, \frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \lambda \text{ (}\lambda\text{ is the separation constant).}$$

Case-I: Let $\lambda > 0$. Take $\lambda = \mu^2$.

$$\therefore X''(x) = \mu^2 X(x), or\ X(x) = (Ae^{\mu x} + Be^{-\mu x}),$$

$$T'(t) = k\mu^2 T(t), or\ T(t) = Ce^{k\mu^2 t}.$$

$$u(x,t) = Ce^{k\mu^2 t}(Ae^{\mu x} + Be^{-\mu x}) = e^{k\mu^2 t}(A'e^{\mu x} + B'e^{-\mu x})[A' = CA, B' = CB].$$

With the B.Cs $u(0,t) = 0 = u(l,t)$,

$$A' + B' = 0, \qquad A'e^{\mu l} + B'e^{-\mu l} = 0.$$

The coefficient determinant of the system is $\begin{vmatrix} 1 & 1 \\ e^{\mu l} & e^{-\mu l} \end{vmatrix}$ which is not equal to zero. Hence, only trivial solution $A' = B' = 0$ exists.

Hence, in this case $u = 0$.

Case-II: Let $\lambda = 0$.

$$\therefore X''(x) = 0, or\ X(x) = (A + Bx),$$

$$T'(t) = 0, T(t) = C.$$

$$\therefore u(x,t) = (A + Bx)C = (A' + B'x) \ \ [A' = CA, B' = CB].$$

Using the B.Cs, we find $A = 0, B = 0$.

Hence, this case also produces trivial solution.

Case-III: Let $\lambda < 0$. Let $\lambda = -\mu^2$.

$$\therefore X''(x) = -\mu^2 X(x), or, X(x) = (A\cos\mu x + B\sin\mu x),$$

$$T'(t) = -k\mu^2 T(t), or\ T(t) = Ce^{-k\mu^2 t}.$$

$$\therefore u(x,t) = Ce^{-k\mu^2 t}(A\cos\mu x + B\sin\mu x) = e^{-k\mu^2 t}(a\cos\mu x + b\sin\mu x)[a = AC, b = BC].$$

Applying the B.Cs $u(0,t) = u(l,t) = 0$, we find

$$a = 0, \qquad b\sin\mu l = 0,$$

$$or, \sin\mu l = 0 = \sin n\pi\ [b \neq 0, \text{otherwise only trivial solution is found}],$$

$$or\ \ \mu_n = \frac{n\pi}{l}, n \in N.$$

$$\therefore u_n(x,t) = b_n e^{-k\mu_n{}^2 t}\sin\ \mu_n x.$$

By principle of superposition,

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} b_n e^{-k\mu_n{}^2 t}\sin\ \mu_n x\ . \dots (1)$$

With the I.C $u(x,0) = f(x), 0 \le x \le l$, we obtain the following result

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\ \mu_n x\,, \qquad \dots (2)$$

(1) is a half range Fourier series. Hence, b_n is given by

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\ \mu_n x\, dx. \dots (3)$$

Non-homogeneous boundary conditions:

a) Consider the heat conduction equation

$$u_t = ku_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$u(0,t) = 0, \qquad u(l,t) = u_0, \qquad t > 0,\ \ [\text{Non} - \text{homogeneous B.Cs}]$$

$$u(x,0) = f(x),\ \ 0 \le x \le l.$$

Let us suppose $u(x,t) = v(x,t) + W(x)$.

Then,

$$v_t = kv_{xx} + kW''(x), \qquad 0 < x < l, \qquad t > 0,$$

$$v(0,t) = -W(0), \qquad v(l,t) = u_0 - W(l), \qquad t > 0,$$

$$v(x,0) = f(x) - W(x), \qquad 0 \le x \le l.$$

Let us again suppose $W(x)$ satisfy the BVP

$$W''(x) = 0, \qquad W(0) = 0, \qquad W(l) = u_0. \;\; \dots (4)$$

Solution of (4) is

$$W(x) = C_1 + C_2 x, C_1, C_2 \text{ are arbitrary constants.}$$

$$\because W(0) = 0, \qquad W(l) = u_0, \qquad C_1 = 0, \qquad C_2 = \frac{u_0}{l}.$$

$$\therefore W(x) = \frac{u_0 x}{l}.$$

$v(x,t)$ then satisfies the PDE

$$\left.\begin{array}{c} v_t = kv_{xx}, \;\; 0 < x < l, \;\; t > 0, \\ v(0,t) = 0, \qquad v(l,t) = 0, \qquad t > 0, \\ v(x,0) = f(x) - \dfrac{u_0 x}{l} = f'(x), \qquad 0 \le x \le l. \end{array}\right\} \;\; \dots (5)$$

Solution of (5) is given by

$$v(x,t) = \sum_{n=0}^{\infty} b_n e^{-k{\mu_n}^2 t} \sin \mu_n x \;\; \dots (6)$$

with

$$b_n = \frac{2}{l}\int_0^l f'(x) \sin \mu_n x \, dx, \qquad \mu_n = \frac{n\pi}{l}. \;\; \dots (7)$$

b) Let the boundary conditions be

$$u_t = ku_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$u(0,t) = 0, \qquad u_x(l,t) = u_0, \qquad t > 0, \;\; [\text{Non}-\text{homogeneous B.Cs}]$$

$$u(x,0) = f(x), \;\; 0 \le x \le l.$$

Let us assume a solution of the form $u(x,t) = v(x,t) + W(x)$.

$$\therefore v_t = kv_{xx} + kW''(x), \qquad 0 < x < l, \qquad t > 0,$$

$$v(0,t) = -W(0), \qquad v_x(l,t) = u_0 - W'(l), \qquad t > 0,$$

$$v(x,0) = f(x) - W(x) = f'(x), \;\; 0 \le x \le l.$$

Let us suppose $W(x)$ is the solution of the BVP

$$W''(x) = 0, \qquad W'(0) = 0, \qquad W'(l) = u_0. \dots (8)$$

Solution of the BVP is $W(x) = u_0 x$.

$v(x,t)$ then satisfies

$$\left.\begin{aligned} v_t &= k v_{xx}, \;\; 0 < x < l, \;\; t > 0, \\ v(0,t) &= 0, \qquad v_x(l,t) = 0, \qquad t > 0, \\ v(x,0) &= f(x) - u_0 x = f'(x), \qquad 0 \le x \le l. \end{aligned}\right\} \dots (9)$$

For the solution $v(x,t)$, let us take $v(x,t) = X(x)T(t)$.

Taking the separation constant $\lambda = -\mu^2 < 0$, we obtain

$$v(x,t) = e^{-k\mu^2 t}(a\cos\mu x + b\sin\mu x).$$

With the B.Cs $v(0,t) = 0, \;\; v_x(l,t) = 0$, we find

$$a = 0, \qquad b\cos\mu l = 0, or \;\; \cos\mu l = 0 \; [\because b \neq 0].$$

$$\therefore \mu_n = \frac{(2n+1)\pi}{l}, n \in N.$$

$$\therefore v_n(x,t) = b_n e^{-k\mu_n{}^2 t}\sin\mu_n x.$$

$$v(x,t) = \sum_{n=0}^{\infty} b_n e^{-k\mu_n{}^2 t}\sin\; \mu_n x \dots (10)$$

Now apply the I.C $v(x,0) = f'(x)$,

$$f'(x) = \sum_{n=0}^{\infty} b_n \sin\; \mu_n x \,, or\; b_n = \frac{2}{l}\int_0^l f'(x)\sin\; \mu_n x\, dx. \dots (11)$$

Uniqueness Theorem:

Let $u(x,t)$ be twice continuously differentiable function of x, t. If $u(x,t)$ satisfies

$$u_t = k u_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

the boundary conditions

$$u(0,t) = u(l,t) = 0, \qquad t > 0,$$

the initial condition

$$u(x,0) = f(x), \qquad 0 \leq x \leq l,$$

then the solution is unique.

Proof: Let there exists two different solutions $u_1(x,t)$ and $u_2(x,t)$. Let $v(x,t) = u_1(x,t) - u_2(x,t)$.

Our objective is to prove $v(x,t) = 0$.

Now, $v(x,t)$ satisfies

$$u_t = ku_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$v(0,t) = v(l,t) = 0, \qquad t \geq 0,$$

$$v(x,0) = 0, \qquad 0 \leq x \leq l.$$

Let us consider the integral

$$J(t) = \frac{1}{2k}\int_0^l v^2\, dx. \dots (12)$$

Differentiating (12) w.r.t t,

$$J'(t) = \frac{1}{2k}\int_0^l 2vv_t\, dx = \frac{1}{k}\int_0^l vv_t\, dx = \frac{1}{k}\int_0^l kvv_{xx}\, dx = \int_0^l vv_{xx}\, dx.$$

Now,

$$\int_0^l vv_{xx}\, dx = v.v_x\Big|_0^l - \int_0^l {v_x}^2\, dx = -\int_0^l {v_x}^2\, dx \leq 0.$$

Since, $v(x,0) = 0$, we have $J(0) = 0$.

$\therefore J(t)$ is non-increasing function of t and $J(t) \leq J(0) = 0$.

However, by definition $J(t) \geq 0, \therefore J(t) = 0$ for all $t \geq 0$.

Hence $v(x,t) = 0$ for all $0 \leq x \leq l,\ t \geq 0$.

Problems:

Solve the following heat conduction problems:

1.

$$u_t = 4u_{xx}, \qquad 0 < x < 1, \qquad t > 0,$$

$$u(0,t) = 0 = u(1,t), \qquad t > 0,$$

$$u(x,0) = x^2(1-x), \;\; 0 \le x \le 1$$

2.

$$u_t = ku_{xx}, \qquad 0 < x < l, \qquad t > 0,$$

$$u(0,t) = 0, \qquad u(l,t) = 1, \qquad t > 0,$$

$$u(x,0) = \sin\frac{\pi x}{2l}, \;\; 0 \le x \le l.$$

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