POSITIONAL NUMBER SYSTEM & BINARY ARITHMETIC

In this system, a set of symbols are used to form a number. These symbols are known as digits, total number of which is called base (or radix) of the system. The weight of each digit in a number depends on its relative position within the number and is some integral power of the base. The following table provides us knowledge about different number systems.

Number system	Base	Symbols used
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9
Binary	2	0, 1
Octal	8	0, 1, 2, 3, 4, 5, 6, 7
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Any number $W_{k-1}\, W_{k-2} \dots W_0.\ W_{-1}\, W_{-2} \dots W_{-l}$ where $W_{k-1}, W_{k-2}, \dots, W_0$ are the digits appearing before the decimal point and $W_{-1}, W_{-2}, \dots, W_{-l}$ are digits appearing after the decimal point, can be expanded using the features of this number system, as

$$W_{k-1}\, W_{k-2} \dots W_0.\ W_{-1}\, W_{-2} \dots W_{-l}$$

$$= W_{k-1} \times b^{k-1} + W_{k-2} \times b^{k-2} + \cdots + W_0 \times b^0 + W_{-1} \times b^{-1} + \cdots$$

$$+ W_{-l} \times b^{-l} \qquad ..(1),$$

b being the base of system. (1) gives decimal equivalent of a number in any system.

Binary Number system

Binary number system allows only two symbols 0 & 1to form a number. These symbols (0&1) are known as **bits.** The base or radix of this number system is 2. The bit at the extreme left of a binary number has the highest positional value and is called Most Significant Bit (MSB). Similarly, the bit at the extreme right position of a given binary number has the least positional value and is known as Least Significant Bit (LSB).

Conversion

Binary to Decimal:

A n-bit binary number of the form $a_{n-1}\, a_{n-2} \dots a_0.\ a_{-1}\, a_{-2} \dots a_{-m}$ where each a_i $(i = -m, \dots, -1, 0, 1, \dots, n-1)$ is either 0 or 1 has the value

$a_{n-1} \times 2^{n-1} + a_{n-2} \times 2^{n-2} + \cdots + a_0 \times 2^0 + a_{-1} \times 2^{-1} + \cdots + a_{-m} \times 2^{-m}$ in decimal system.

i) Convert 1101_2 to equivalent decimal number.

$1101_2 = 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0 = 8 + 4 + 0 + 1 = 13_{10}$

ii) Convert 1011.1101_2 to equivalent decimal number.

$$1011.1101_2 = 1\times 2^3 + 0\times 2^2 + 1\times 2^1 + 1\times 2^0 + 1\times 2^{-1} + 1\times 2^{-2} + 0\times 2^{-3} + 1\times 2^{-4}$$
$$= 8 + 2 + 1 + 0.5 + 0.25 + 0.0625 = 11.8125_{10}$$

Decimal to Binary:

A whole decimal number or the integer part of a fractional number is converted into equivalent binary by successive division by 2, writing down each quotient and its remainder. The equivalent binary number is just the collection of the remainder in reverse order. The method to find binary equivalent of the fractional part is quite different. This method involves multiplication of the fraction successively by 2. In each step, the whole number (integer part) obtained (either 0 or 1) after multiplication is noted and the new fraction part (if any) obtained is again used for multiplication. The process continues till either the fraction reduces to 0 or the desired number of binary places is obtained. The binary corresponding to the original fraction is thus found by arranging the integer parts in forward direction.

i) Convert 103_{10} to its equivalent binary number.

Divisor	Quotient	remainder
2	103	
2	51	1
2	25	1
2	12	1
2	6	0
2	3	0
2	1	1
	0	1 ↑

Thus the equivalent binary number is 1100111_2

ii) Convert 24.25_{10} to its equivalent binary number.

Divisor	Quotient	remainder
2	24	
2	12	0
2	6	0
2	3	0
2	1	1
	0	1 ↑

$\therefore 24_{10} = 11000_2$

Multiplication	Integer part	Fraction
$.25 \times 2 = .5$	0	.5
$.5 \times 2 = 1.0$	1	0

$\therefore .25_{10} = .01_2$

Thus the result is $24.25_{10} = 11000.01_2$

Octal Number System

This number system allows eight digits $0, 1, 2, 3, 4, 5, 6, 7$ to form a number. The base of this system is 8. As there exist only eight digits in octal number system so 3 bits are sufficient to represent any octal digit in binary number system.

Conversion

<u>Octal to Decimal</u>:

This conversion has been done using the rule (1).

i) Convert 2057.45_8 to its equivalent decimal number.

$$2057.45_8 = 2 \times 8^3 + 0 \times 8^2 + 5 \times 8^1 + 7 \times 8^0 + 4 \times 8^{-1} + 5 \times 8^{-2}$$
$$= 1024 + 0 + 40 + 7 + .5 + .078125 = 1071.578125_{10}$$

<u>Decimal to Octal</u>:

The conversion from decimal to octal can be done using the same method as that of decimal to binary. The only difference is that here successive division and multiplication are done by 8 instead of 2.

i) Convert 520.375_{10} to its equivalent octal form.

		remainder
8	520	
8	65	0
8	8	1
8	1	0
	0	1

↑

$\therefore 520_{10} = 1010_8$

Multiplication	Integer part	Fraction
$.375 \times 8 = 3.0$	3	0

$\therefore .375_{10} = .3_8$
$\therefore 520.375_{10} = 1010.3_8$

<u>Binary to Octal</u>:

The integer part of the binary number is arranged into groups of 3 bits beginning at the binary point and proceeding to the left. If the number of bits is not a multiple of 3, we add necessary number of zeros to the left of MSB. On the other hand, fractional part of the number is separated into 3-bit groups beginning at the binary point and proceeding towards right. In case of fractional part if the number of bits is not a multiple of 3, we add necessary number of zeros to the right of LSB. The following table gives equivalent binary number in 3 bit corresponding to each octal digit.

Decimal Number	Octal Number	Binary Number (3 bit)
0	0	000
1	1	001

2	2	010
3	3	011
4	4	100
5	5	101
6	6	110
7	7	111

i) Convert 111001_2 to equivalent octal number.
$111001 = \overline{111}\,\underline{001} = 71_8$

ii) Convert 11110.101_2 to equivalent octal number
$11110.101 = \overline{011}\underline{110}.\overline{101} = 36.5_8$

Octal to Binary:

Octal to binary conversion can be done by replacing each digit by its 3-bit binary equivalent.

i) Convert 637.41_8 into binary form.
$637.41_8 = 110\ \ 011\ \ 111\ \ .\ \ 100\ \ \ 001 = 110011111.100001_2$

Hexadecimal Number System

This number system has a base 16 and uses 16 digits 0, 1, 2, 3, 4 ,5, 6, 7, 8, 9, A, B, C, D, E, F to construct a number. The symbols A through F represent the equivalent decimal numbers 10 through 15. As there are only 16 digits in hexadecimal number system so 4 bits are sufficient to represent any hexadecimal digit in binary.

Conversion

Hexadecimal to Decimal:
This conversion has been done using the rule (1).

i) Convert $1AF_{16}$ into decimal form.
$1AF_{16} = 1 \times 16^2\ +\ 10 \times 16^1\ +\ 15 \times 16^0 = 256 + 160 + 15 = 431_{10}$

ii) Convert $\mathrm{BCD.EF}_{16}$ into decimal form.
$\mathrm{BCD.EF}_{16} = 11 \times 16^2\ +\ 12 \times 16^1\ +\ 13 \times 16^0 + 14 \times 16^{-1}\ +\ 15 \times 16^{-2}$
$= 3021.933594_{10}$

Decimal to Hexadecimal:

The conversion from decimal to Hexadecimal can be done using the same method as that of decimal to binary. The only difference is that here successive division and multiplication are done by 16 instead of 2.

i) Convert 65025.78125_{10} into hexadecimal form.

16	65025	remainder
16	4064	1

16	254	0
16	15	14(=E)
	0	15(=F)

$65025_{10} = \text{FE01}_{16}$

Multiplication	Integer part	Fraction
$.78125 \times 16 = 12.5$	12	.5
$.5 \times 16 = 8$	8	0

$.78125_{10} = .\text{C8}_{16}$

$65025.78125_{10} = \text{FE01}.\text{C8}_{16}$

Binary to Hexadecimal:

The integer part of the binary number is separated into groups of 4 bits beginning at the binary point and proceeding to the left. If the number of bits in this part is not a multiple of 4, add necessary number of zeros to the left of MSB. For the fractional part the same procedure is followed and we have to proceed towards right. In case of fractional part if the number of bits is not a multiple of 4, add necessary number of zeros to the right of LSB. The following table gives equivalent binary number in 4 bits corresponding to each hexadecimal digit.

Decimal Number	**Hexadecimal Number**	**Binary Number (4 bit)**
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
10(=A)	10	1010
11(=B)	11	1011
12(=C)	12	1100
13(=D)	13	1101
14(=E)	14	1110
15(=F)	15	1111

i) Convert 11010101000.1111010111_2 to equivalent hexadecimal number.

$11010101000.1111010111_2 = \underline{110}\;\overline{1010}\;\underline{1000}.\overline{1111}\;\underline{0101}\;\overline{11}$

$= 0110\;\;1010\;\;1000.1111\;\;0101\;\;1100 = 6\text{A}8.\text{F}5\text{C}_{16}$

Hexadecimal to Binary:

This conversion is just the reverse of above process.

i) Convert $2AB.F6_{16}$ into binary number.

$2AB.F6_{16} = 0010\ \ 1010\ \ 1011.1111\ \ 0110 = 1010101011.1111011_2$

Hexadecimal to Octal:

Hexadecimal number is first converted to binary number and then the binary number is converted to octal number.
e.g.

i) Convert $FB.CD_{16}$ into octal number.

$FB.CD_{16} = 1111\ \ 1011.1100\ \ 1101 = 11111011.11001101_2$

$= \underline{011}\ \overline{111}\ \ \underline{011}.\overline{110}\ \ \underline{011}\ \ \overline{010}$

$= 373.632_8$

Octal to Hexadecimal:

Octal number is first converted to binary number and then the binary number is converted to hexadecimal number

i) Convert 73.24_8 into hexadecimal number.

$73.24_8 = 111\ \ 011.010\ \ 100 = 111011.0101_2 = \overline{0011}\ \ \underline{1011}.\overline{0101} = 3B.5_{16}$

Binary Arithmetic

1. Binary Addition:

The following rules enable us to add binary numbers:

Rule 1 0 + 0 = 0

Rule 2 0 + 1 = 1

Rule 3 1 + 0 = 1

Rule 4 1 + 1 = $\mathbf{1}^c0$ i.e. one plus one equals to zero with **carry one** to the next column on the left.

Rule 5 1 + 1 + 1 =$\mathbf{1}^c1$ i.e. one plus one plus one equals to one with **carry one** to the next column on the left.

Examples:

i) $10001_2 + 01000_2$

C5	C4	C3	C2	C1
1	0	0	0	1
0	1	0	0	0
1	1	0	0	1

Explanation: C1: 1 + 0 = 1 (Rule 3)

C2: 0 + 0 = 0 (Rule 1)

C3: 0 + 0 = 0 (Rule 1)

C4: 0 + 1 = 1 (Rule 2)

C5: 1 + 0 = 0 (Rule 3)

ii) $100.11_2 + 101.01_2$

C6	C5	C4	C3	C2	C1
	1	0	0 .	1	1
	1	0	1 .	0	1
1	0	1	0 .	0	0

Explanation: C1: 1 + 1 = 0 with carry 1 (Rule 4)

C2: 1 + 0 + 1(carry from C1) = 0 with carry 1 (Rule 3 & 4)

C3: 0 + 1 +1 (carry from C2) = 0 (Rule 2 & 4)

C4: 0 + 0 + 1 (carry from C3) = 1 (Rule 1 & 2)

C5: 1 + 1 = 0 with carry 1(Rule 4)

C6: 1 (carry from C5)

2. Binary Subtraction :

The following rules enable us to subtract binary numbers.

Rule 1 0 - 0 = 0

Rule 2 0 - 1 = 1^b1 i.e. zero plus one equals to one with **borrow one** from the next column on the **left.**

Rule 3 1 - 0 = 1

Rule 4 1 - 1 = 0

Examples:

i) $111_2 - 101_2$

C3	C2	C1
1	1	1

```
 - 1   0   1
 -----------
   0   1   0
```

Explanation: C1: 1 - 1 = 0 (Rule 4)

C2: 1 - 0 = 0 (Rule 3)

C3: 1- 1 = 0 (Rule 4)

ii) $1110010_2 - 1100100_2$

```
   C7  C6  C5  C4  C3  C2  C1
   1   1   1   0   0   1   0
 - 1   1   0   0   1   0   0
 ---------------------------
   0   0   0   1   1   1   0
```

Explanation: C1: 0 - 0 = 0 (Rule 1)

C2: 1 - 0 = 1 (Rule 3)

C3: 0 - 1 = 1 with borrow 1 (Rule 2)

C4: 0 -0 - 1 (borrow from C3) = 1 with borrow 1 (Rule 1 & 2)

C5: 1 – 0 – 1(borrow from C4) = 0 (Rule 3 & 4)

C6: 1 – 1 = 0 (Rule 4)

C7: 1 – 1 = 0 (Rule 4)

3. 1's Complement and 2's Complement of a Binary Number:

1's Complement of a binary number is obtained by replacing each 1 in the number by 0 and each 0 by 1.

e.g.

Binary Number	1's Complement
1101	0010
1000	0111
10101011	01010100

2's Complement of a binary number is obtained by adding 1 to LSB of the 1's Complement of a number. e.g.

Binary Number	1's Complement	2's Complement
0111	1000	1000

		+ 1 1001
00001000	11110111	11110111 + 1 11111000
101011	010100	010100 +1 010101

4. Binary Subtraction using 1's Complement and 2's Complement:

Using 1's Complement:

The following steps are to be followed:

Step I: Determine 1's Complement of the subtrahend.

Step II: Add 1's Complement thus obtained to the minuend.

Step III: Check whether there is any carry over generated from the leftmost column. If there is any, then add this carry to the LSB of the sum obtained in the step II.

Step IV: If there is no carry over from leftmost column, then 1's Complement of the sum obtained in the step II will be the result and it will be negative.

e.g.

i) $11000_2 - 10101_2$

First we have to find 1's Complement of the subtrahend 10101_2. 1's Complement of 10101 is 01010.

Now find $11000_2 + 01010_2$

	C5	C4	C3	C2	C1
	1	1	0	0	0
+	0	1	0	1	0
carry → 1	0	0	0	1	0
					1
	0	0	0	1	1

The answer is 00011_2

ii) $0110_2 - 1001_2$

First we have to find 1's Complement of the subtrahend 1001_2. 1's Complement of 1001 is 0110.

Now find $0110_2 + 0110_2$

	C4	C3	C2	C1
	0	1	1	0
+	0	1	1	0
	1	1	0	0

Since there is no carry 1 from the leftmost column, the result will be just 1's Complement of the sum thus obtained. Therefore result is 0011_2 with a negative sign.

iii) $10111.01_2 - 1101.11_2$

Before applying 1's Complement method make it sure that both the numbers contain same number of digits before and after decimal point. If there be any mismatch put 0s. Thus here the number 1101.11 is transformed to 01101.11

First we have to find 1's Complement of the subtrahend 01101.11.

1's Complement of 01101.11 is 10010.00.

Now find $10111.01_2 + 10010.00_2$

	C7	C6	C5	C4	C3		C2	C1
	1	0	1	1	1	.	0	1
	1	0	0	1	0	.	0	0
carry 1	0	1	0	0	1	.	0	1

(Carry 1 is to be added on to the sum obtained)

+								1
	0	1	0	0	1	.	1	0

Answer = 1001.10_2

Using 2's Complement:

The following steps are to be followed:

Step I: Find 2's Complement of the subtrahend.

Step II: Add 2's Complement thus obtained to the minuend.

Step III: If there is any carry over 1 generated from the leftmost column, then drop this. The sum obtained in the step II will be the answer and it will be positive.

Step IV: On the other hand if the carry is 0 from leftmost column, then result will be the 2's complement of the sum with a negative sign.

e.g.

i) $110.011_2 - 101.010_2$

First we have to find 2's Complement of the subtrahend 101.101_2.

1's Complement of 101.010 is 010.101. Therefore 2's Complement is (010.101+1) i.e. 010.110

Now find $110.011_2 + 010.011_2$

		C6	C5	C4		C3	C2	C1
		1	1	0	.	0	1	1
+		0	1	0	.	1	1	0
carry→	1	0	0	1	.	0	0	1

Since there is a carry 1 from C6, the result of the subtraction is 001.001_2 i.e. 1.001_2

ii) $01111_2 - 11111_2$

First we have to find 2's Complement of the subtrahend 11111_2. 1's Complement of 11111 is 00000. Thus 2's Complement is $(00000 + 1)$ i.e. 00001

Now find $01111_2 + 00001_2$

	C5	C4	C3	C2	C1
	0	1	1	1	1
+	0	0	0	0	1
	1	0	0	0	0

Since there is no carry from C5, the result of subtraction will be just 2's Complement of the sum obtained with a negative sign.

2's Complement of 10000_2 is $(01111 + 1)$ i.e. 10000. Hence answer is -10000_2

5. Binary multiplication:

The following rules are to be remembered in order to perform binary multiplication:

Rule I $\quad 0 \times 0 = 0$

Rule II $0 \times 1 = 0$

Rule III $1 \times 0 = 0$

Rule IV $1 \times 1 = 1$

e.g.

i) Multiply 1010_2 by 101_2

```
     1 0 1 0
   × 1 0 1
-----------
     1 0 1 0
   0 0 0 0 ×
-----------
   0 1 0 1 0
 1 0 1 0 × ×
-----------
 1 1 0 0 1 0
-----------
```

ii) Multiply 101.1_2 by 101.1_2

```
       1 0 1.1
     × 1 0 1.1
--------------
       1 0 1 1
     1 0 1 1 ×
--------------
   1 0 0 0 0 1
   0 0 0 0 × ×
--------------
   1 0 0 0 0 1
 1 0 1 1 × × ×
--------------
 1 1 1 1 0.0 1
--------------
```

6. Binary division:

The rules for binary division are given below:

Rule I $0 \div 1 = 0$

Rule II $1 \div 1 = 1$

Rule III $(0 \div 0)\&(1 \div 0)$ both are meaning less.

e.g.

i) $11001 \div 101$

```
101) 11001 (101
      101
      ----
        101
        101
      ----
```

Remainder is 0 and quotient is101.

ii) $11101.01 \div 1100$

```
1100) 11101.01(10.0111
      1100
      ------
        10101
         1100
      ------
         10010
          1100
      ------
           1100
           1100
        ------
```

The remainder is 0 and quotient is 10.0111.