WEAK LAW OF LARGE NUMBERS
AND
ITS RELATED PROBLEMS

Presented

by

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STATEMENT: Let $x_1, x_2, x_3, \ldots\ldots$ be a sequence of random variables having expectations $\mu_1, \mu_2, \mu_3, \ldots\ldots$ Further, let $V_n = Var(x_1 + x_2 + \ldots\ldots\ldots + x_n)$. If $\frac{V_n}{n^2} \to 0$ as $n \to \infty$ then given any two (+)ve quantities ε and η however, small, we can find an n_0, depending on ε and η such that

$$P\left[\left|\frac{x_1 + x_2 + \ldots\ldots\ldots + x_n}{n} - \frac{\mu_1 + \mu_2 + \ldots\ldots + \mu_n}{n}\right| \le \varepsilon\right] > 1 - \eta \ \forall n \ge n_0.$$

PROOF:

$$Since,\ E\left(\frac{x_1+x_2+......+x_n}{n}\right)=\frac{\mu_1+\mu_2+.......+\mu_n}{n}$$

$$and\ Var\left(\frac{x_1+x_2+......+x_n}{n}\right)=\frac{V_n}{n^2}$$

$U \sin g\ Chebyshev's\ inequality$

$$P\left[\left|\frac{x_1+x_2+.......+x_n}{n}-\frac{\mu_1+\mu_2+......+\mu_n}{n}\right|\le t\frac{\sqrt{V_n}}{n}\right]>1-\frac{1}{t^2}$$

$for\ any\ (+)ve\ t.\ Choo\sin g\ t\ in\ such\ a\ way\ that\ \dfrac{t\sqrt{V_n}}{n}=\varepsilon.$

$$Therefore,\ for\ any\ \varepsilon > 0,$$

$$P\left[\left|\frac{x_1 + x_2 + \ldots\ldots. + x_n}{n} - \frac{\mu_1 + \mu_2 + \ldots\ldots + \mu_n}{n}\right| \le \varepsilon\right] > 1 - \frac{V_n}{n^2\varepsilon^2}$$

$$Since,\ \frac{V_n}{n^2} \to 0\ as\ n \to \infty,\ given\ \eta.\varepsilon^2 > 0\ however\ small,$$

$$an\ n_0\ is\ found\ depending\ on\ \eta.\varepsilon^2,\ such\ that\ \frac{V_n}{n^2} < \eta\varepsilon^2, \forall n \ge n_0.$$

$$For\ such\ an\ n_0,\quad P\left[\left|\frac{x_1 + x_2 + \ldots\ldots. + x_n}{n} - \frac{\mu_1 + \mu_2 + \ldots\ldots + \mu_n}{n}\right| \le \varepsilon\right] > 1 - \eta$$

$$whenever\ n \ge n_0.$$

Bernoulli's Law of large numbers

STATEMENT: Let there be n trials of an event, each trial resulting in a success or failure. If X is the number of successes in n trials with constant probability p of success for each trial, then E(X)=np and Var(X)=npq, q=1-p. The variable X / n represents the proportion of successes or the relative frequency of successes and

$$E\left(\frac{X}{n}\right) = p \text{ and } Var\left(\frac{X}{n}\right) = \frac{Var(X)}{n^2} = \frac{pq}{n}.$$

$$Then,\ P\left[\left|\frac{X}{n} - p\right| < \varepsilon\right] \to 1 \text{ as } n \to \infty \quad \Rightarrow P\left[\left|\frac{X}{n} - p\right| \geq \varepsilon\right] \to 0 \text{ as } n \to \infty$$

for any assigned $\varepsilon > 0$. *This implies that* $\frac{X}{n}$ *converges in probability to* p *as* $n \to \infty$.

PROOF: $\textit{Applying Chebyshev's inequality to the variable } \frac{X}{n} \textit{ for any } \varepsilon > 0,$

$$\textit{it is found that, } P\left[\left|\frac{X}{n} - E\left(\frac{X}{n}\right)\right| \geq \varepsilon\right] \leq \frac{Var\left(\frac{X}{n}\right)}{\varepsilon^2} = \frac{pq}{n\varepsilon^2} \leq \frac{1}{4n\varepsilon^2}$$

$$\textit{Since, the } \max \textit{ value of } pq \textit{ is at } p = q = \frac{1}{2}, \textit{i.e.} \max(pq) = \frac{1}{4} \textit{ i.e. } pq \leq \frac{1}{4}.$$

$$\textit{Since, } \varepsilon \textit{ is arbitrary therefore } P\left[\left|\frac{X}{n} - p\right| \geq \varepsilon\right] \to 0 \textit{ as } n \to \infty$$

$$\Rightarrow P\left[\left|\frac{X}{n} - p\right| < \varepsilon\right] \to 0 \textit{ as } n \to \infty$$

Central Limit Theorem

If X_i $(i = 1, 2, \ldots\ldots\ldots, n)$ be independent random variables such that $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$, then under certain very general conditions, the random variable $S_n = X_1 + X_2 + \ldots\ldots + X_n$ is assymptotically normal with mean μ and standard deviation σ. Where, $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

De – Moivre's Laplace theorem

$$If\ \ X_i = 1\ with\ \ probability\ \ p$$

$$= 0\ with\ \ probability\ q = 1 - p\ (i = 1,2,.........,n)$$

$$then\ the\ distribution\ of\ \ the\ random\ variable$$

$$S_n = X_1 + X_2 + ...+ X_n,\ where\ \ X_i's\ are\ independent,$$

$$is\ assymptotically\ normal\ as\ n \to \infty.$$

Lindeberg Levy Central Limit theorem

If X_i $(i = 1, 2, \ldots\ldots\ldots, n)$ be independent random variables such that $E(X_i) = \mu_1$ and $Var(X_i) = \sigma_1^2$, then the sum $S_n = X_1 + X_2 + \ldots\ldots + X_n$ is assymptotically normal with mean $\mu = n\mu$ and variance $\sigma^2 = n\sigma_1^2$.

Liapounoff's Central Limit theorem

If X_i $(i = 1, 2, \ldots\ldots\ldots, n)$ be independent random variables such that $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$. Let, the third order absolute moment, say ρ_i^3 of X_i about its mean exists i.e. $\rho_i^3 = E\left[|X_i - \mu_i|^3\right]$, $i = 1, 2, \ldots\ldots, n$ is finite. Further let $\rho^3 = \sum_{i=1}^{n} \rho_i^3$.

If $\lim n \to \infty \frac{\rho}{\sigma} = 0$, then the sum $X = X_1 + X_2 + \ldots\ldots + X_n$ is assymptotically normal with mean μ and standard deviation σ. Where, $\mu = \sum_{i=1}^{n} \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$.

PROBLEM:

a) Let X_i assume the values i and $(-i)$ with equal probabilities. Show that the law of large numbers can not be applied to the independent variables X_1, X_2,……i.e. X_i's.

b) Let X_i can have only two values i^{α} and $(-i^{\alpha})$ with equal probabilities. Show that the law of large numbers can be applied to the independent variables X_1, X_2,……if $\alpha < \frac{1}{2}$.

PROOF:

$$a)\ Here,\ P[X_i = i] = \frac{1}{2} = P[X_i = -i]$$

$$E(X_i) = i.\frac{1}{2} - i.\frac{1}{2} = 0\ \forall i;\ i = 1,2,..........$$

$$V(X_i) = E(X_i^{\,2}) = i^2.\frac{1}{2} + (-i)^2.\frac{1}{2} = i^2\ \forall i;\ i = 1,2,..........$$

$$B_n = V(\sum_{i=1}^{n} X_i) = 1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore \frac{B_n}{n^2} \to \infty\ as\ n \to \infty.$$

Hence, law of large numbers does not hold.

$$b)\ Here,\ P[X_i = i^{\alpha}] = \frac{1}{2} = P[X_i = -i^{\alpha}]$$

$$E(X_i) = i^{\alpha}.\frac{1}{2} - i^{\alpha}.\frac{1}{2} = 0\ \forall i;\ i = 1,2,..........$$

$$V(X_i) = E(X_i^{\,2}) = i^{2\alpha}.\frac{1}{2} + (-i^{\alpha})^2.\frac{1}{2} = i^{2\alpha}\ \ \forall i;\ i = 1,2,..........$$

$$B_n = V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} i^{2\alpha} = 1^{2\alpha} + 2^{2\alpha} + ...+ n^{2\alpha} \simeq \int_0^n x^{2\alpha}dx$$

From Euler Maclaurin's formula.

$$\cong \left[\frac{x^{2\alpha+1}}{2\alpha+1}\right]_0^n = \frac{n^{2\alpha+1}}{2\alpha+1}$$

$$\therefore \frac{B_n}{n^2} = \frac{n^{2\alpha-1}}{2\alpha+1} \to 0\ if\ \ 2\alpha - 1 < 0 \Rightarrow \alpha < \frac{1}{2}.\ Hence,\ the\ result.$$

PROBLEM:

Let $\{X_k\}$ be mutually independent and identically distributed random variables with mean μ and finite variance. If $S_n = X_1 + X_2 + \ldots\ldots + X_n$, prove that the law of large numbers does not hold for the sequence $\{S_n\}$

PROOF:

The variables now are $S_1, S_2, \ldots\ldots\ldots, S_n.$

$$B_n = V(S_1 + S_2 + \ldots\ldots\ldots\ldots + S_n)$$

$$= V\{X_1 + (X_1 + X_2) + (X_1 + X_2 + X_3) + \ldots\ldots + (X_1 + X_2 + \ldots\ldots\ldots\ldots + X_n)\}$$

$$= V\{nX_1 + (n-1)X_2 + (n-2)X_3 + \ldots\ldots + X_n\}$$

$$= n^2 V(X_1) + (n-1)^2 V(X_2) + (n-2)^2 V(X_3) + \ldots\ldots + V(X_n)$$

$$Let, V(X_i) = \sigma^2 \ \forall i. \quad \therefore B_n = (1^2 + 2^2 + \ldots\ldots\ldots\ldots + n^2).\sigma^2$$

$$\boxed{\therefore \frac{B_n}{n^2} = \frac{(n+1)(2n+1)}{6n}\sigma^2 \to \infty \text{ as } n \to \infty.}$$

Hence, law of large numbers does not hold for the sequence $\{S_n\}.$

PROBLEM:

Examine whether the law of large number holds for the sequence $\{X_k\}$ *of independent random variables defined as follows*

$$P[X_k = \pm 2^k] = 2^{-(2k+1)}$$

$$P[X_k = 0] = 1 - 2^{-2k}$$

PROOF:

$$E(X_k) = 2^k . 2^{-(2k+1)} + (-2)^k . 2^{-(2k+1)} + 0.(1 - 2^{-2k}) = 0$$

$$E(X_k^2) = 2^{2k} . 2^{-(2k+1)} + (-2)^{2k} . 2^{-(2k+1)} + 0.(1 - 2^{-2k}) = 1$$

$$V(X_k) = E(X_k^2) - E^2(X_k) = 1.$$

$$B_n = V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} V(X_i) = n. \quad \because X_i\text{'s are independent}, i = 1,2,\ldots\ldots, n$$

$$\therefore \frac{B_n}{n^2} = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Hence, law of large number holds for the sequence of r.v. $\{X_k\}$.

Thank you…….