

POISSON DISTRIBUTION

**Presented
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CONTENTS:

- Poisson distribution as a limiting case of Binomial distribution.
- Probability mass function of Poisson Distribution.
- Real life situation of Poisson distribution.
- Moments of Poisson distribution.
- Coefficient of skewness and kurtosis.
- Mode
- Recurrence relation for moments.
- Mean deviation about mean.
- Fitting of Poisson distribution.
- Related problems.

LIMITING FORM OF BINOMIAL DISTRIBUTION IS POISSON

Let $X \sim Bin(n, p)$.

So, $f(x) = P(X = x) = n_{c_x} p^x q^{n-x}; x = 0, 1, 2, \dots, n; 0 < p < 1,$

$$p + q = 1$$

$$\therefore f(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} = \frac{n(n-1)(n-2)\dots(n-x+1)(n-x)!}{x!(n-x)!} p^x q^{n-x}$$

Let $np = \lambda$. So, $p = \frac{\lambda}{n}$. Now, $p \rightarrow 0$ as $n \rightarrow \infty$.

$$\lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} \left[\frac{\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \cdots \frac{n-(x-1)}{n}}{x!} \cdot (np)^x (1-p)^{n-x} \right]$$

$$= \frac{\lambda^x}{x!} \lim_{n \rightarrow \infty} \left[\frac{\left(1 - \frac{\lambda}{n}\right)^n}{\left(1 - \frac{\lambda}{n}\right)^x} \right] = \frac{e^{-\lambda} \lambda^x}{x!}, \quad \lim_{n \rightarrow \infty} \left[1 - \frac{\lambda}{n}\right]^n = e^{-\lambda}$$

$$\lim_{n \rightarrow \infty} \left[1 - \frac{\lambda}{n}\right]^x = 1$$

PROBABILITY – MASS FUNCTION OF POISSON DISTRIBUTION

If a random variable X follows Poisson distribution with parameter λ , then probability mass function of X is given by

$$f(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0, 1, 2, \dots; \lambda > 0$$
$$= 0 \quad , elsewhere.$$

REAL LIFE SITUATION OF POISSON DISTRIBUTION

- ❖ Number of mistakes in a typed page.
- ❖ Number of defects in the insulation of a fifty metre length of wire.
- ❖ Number of cars parked at a place in an hour, say between 10:00a.m. and 11:00 a.m.
- ❖ Number of suicides in a certain period in a city or town.

MOMENTS OF POISSON DISTRIBUTION

$$\mu'_r = E(X^r) = \sum_{x=0}^{\infty} x^r f(x) = \sum_{x=0}^{\infty} x^r \frac{e^{-\lambda} \lambda^x}{x!}.$$

$$\mu'_1 = E(X) = \sum_{x=0}^{\infty} x \cdot f(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \cdot \lambda \cdot \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= e^{-\lambda} \cdot \lambda \cdot \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots \right] = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

$$\mu'_2 = E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot f(x) = \sum_{x=0}^{\infty} \{x(x-1) + x\} \cdot \frac{e^{-\lambda} \lambda^x}{x!}.$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^{x-2} \lambda^2}{x(x-1)(x-2)!} + E(X)$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu'_3 = E(X^3) = \sum_{x=0}^{\infty} x^3 \cdot f(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \cdot \frac{e^{-\lambda} \lambda^x}{x!}.$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \cdot \frac{e^{-\lambda} \lambda^3 \lambda^{x-3}}{x(x-1)(x-2)(x-3)!} + 3\lambda^2 + \lambda.$$

$$= e^{-\lambda} \lambda^3 \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} + 3\lambda^2 + \lambda$$

$$= e^{-\lambda} \lambda^3 e^{\lambda} + 3\lambda^2 + \lambda$$

$$= \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu'_4 = E(X^4) = \sum_{x=0}^{\infty} x^4 \cdot f(x)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1)$$

$$+ x\} \cdot \frac{e^{-\lambda} \lambda^x}{x!}.$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3) \frac{e^{-\lambda} \lambda^x}{x!} + 6 \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$+ 7 \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\begin{aligned}
&= e^{-\lambda} \lambda^4 \sum_{x=4}^{\infty} \frac{\lambda^{x-4}}{(x-4)!} + 6e^{-\lambda} \lambda^3 \sum_{x=3}^{\infty} \frac{\lambda^{x-3}}{(x-3)!} \\
&\quad + 7e^{-\lambda} \lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda \\
&= e^{-\lambda} \lambda^4 e^{\lambda} + 6e^{-\lambda} \lambda^3 e^{\lambda} + 7e^{-\lambda} \lambda^2 e^{\lambda} + \lambda \\
&= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda
\end{aligned}$$

$$\mu_2 = \mu'_2 - \mu'^2_1 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

$$\begin{aligned}
\mu_3 &= \mu'_3 - 3\mu'_2\mu'_1 + 2\mu'^3_1 \\
&= \lambda^3 + 3\lambda^2 + \lambda - 3(\lambda^2 + \lambda)\lambda + 2\lambda^3 \\
&= \lambda^3 + 3\lambda^2 + \lambda - 3\lambda^3 - 3\lambda^2 + 2\lambda^3 \\
&= \lambda \\
\\
\mu_4 &= \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1 \\
&= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) \\
&\quad + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4 \\
&= 3\lambda^2 + \lambda
\end{aligned}$$

COEFFICIENT OF SKEWNESS AND KURTOSIS

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \quad \text{and} \quad \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

$$\text{Also, } \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda} \quad \text{and} \quad \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}$$
$$, \lambda > 0$$

Hence, the Poisson distribution is positively skewed and leptokurtic.

MODE

$$\begin{aligned}\frac{f(x)}{f(x-1)} &= \frac{\frac{e^{-\lambda} \lambda^x}{x!}}{\frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!}} \\&= \frac{e^{-\lambda} \lambda^x}{x!} \cdot \frac{(x-1)!}{e^{-\lambda} \lambda^{x-1}} \\&= \frac{\lambda}{x}, \quad x = 1, 2, 3, \dots\end{aligned}$$

Case I:

Let λ be not an integer. $\lambda = m + f$, where m is the integral part and f is the fractional part.

$$\therefore \frac{f(x)}{f(x-1)} = \frac{m+f}{x}$$

So, $\frac{f(x)}{f(x-1)} > 1$; when, $x = 1, 2, \dots, m$

Then, $f(0) < f(1) < \dots < f(m)$

$\frac{f(x)}{f(x-1)} \neq 1$; since, x takes integral values only.

$\frac{f(x)}{f(x-1)} < 1$; when, $x = m+1, m+2, \dots$

Then, $f(m) > f(m+1) > \dots$

Combining above three results, it is seen that
 $f(0) < f(1) < \dots < f(m) > f(m+1) > \dots$

So, the distribution is unimodal and mode is m.

Case II:

Let λ be an integer and $\lambda = m'$

$$\therefore \frac{f(x)}{f(x-1)} = \frac{m'}{x}$$

$\frac{f(x)}{f(x-1)} > 1$; when, $x = 1, 2, \dots, (m' - 1)$.

Then, $f(0) < f(1) < \dots < f(m' - 1)$

$\frac{f(x)}{f(x-1)} = 1$; when, $x = m'$. Then, $f(m' - 1) = f(m')$.

$\frac{f(x)}{f(x-1)} < 1$; when, $x = (m' + 1), (m' + 2), \dots$

Then, $f(m') > f(m' + 1) > f(m' + 2) > \dots$

Combining above three results, it is seen that

$f(0) < f(1) < \dots < f(m' - 1) = f(m') >$
 $f(m' + 1) > f(m' + 2) > \dots$

So, the distribution is bi-modal and modes are

$(m' - 1)$ and m' .

RECURRENCE RELATION FOR MOMENTS

Statement:

$$\mu_{r+1} = r \cdot \lambda \cdot \mu_{r-1} + \lambda \cdot \frac{d\mu_r}{d\lambda}$$

Proof:

$$\mu_r = E(X - \lambda)^r = \sum_{x=0}^{\infty} (x - \lambda)^r \frac{e^{-\lambda} \lambda^x}{x!}$$

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} \frac{1}{x!} [r(x - \lambda)^{r-1} (-1)e^{-\lambda} \lambda^x + (x - \lambda)^r$$

$$\{-e^{-\lambda} \lambda^x + x \lambda^{x-1} e^{-\lambda}\}]$$

$$= -r \sum_{x=0}^{\infty} (x - \lambda)^{r-1} \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} (x - \lambda)^r \frac{e^{-\lambda} \lambda^x}{x!} \left\{ -1 + \frac{x}{\lambda} \right\}$$

$$= -r\mu_{r-1} + \frac{1}{\lambda} \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda} \lambda^x}{x!} \{x-\lambda\} = -r\mu_{r-1} + \frac{1}{\lambda} \mu_{r+1}$$

$$\therefore \mu_{r+1} = \lambda [r\mu_{r-1} + \frac{d\mu_r}{d\lambda}]$$

$$\mu'_1 = \lambda = Mean$$

$$\mu_2 = \lambda [1.\mu_0 + \frac{d\mu_1}{d\lambda}] = \lambda$$

$$\mu_3 = \lambda [2\mu_1 + \frac{d\mu_2}{d\lambda}] = \lambda$$

$$\mu_4 = \lambda [3\mu_2 + \frac{d\mu_3}{d\lambda}] = \lambda [3\lambda + 1] = 3\lambda^2 + \lambda$$

MEAN DEVIATION ABOUT MEAN

Let $X \sim P(\lambda)$ and $\lambda = 1$.

$$\therefore P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-1}}{x!}; x = 0, 1, 2, \dots$$

Mean deviation about mean = $E[|X - E(X)|]$

$$\begin{aligned} &= \sum_{x=0}^{\infty} |x - 1| \frac{e^{-1}}{x!} = e^{-1} \left[1 + \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots \right] \\ &= e^{-1} \left[1 + \left(1 - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \dots \right] \end{aligned}$$

$$= \frac{2}{e} = \frac{2}{e} \cdot \sigma$$

MEAN DEVIATION ABOUT MEAN (λ)

$$MD_{\mu} = E[|X - \lambda|] = \sum_{x=0}^{\infty} |x - \lambda| e^{-\lambda} \cdot \frac{\lambda^x}{x!}$$

$$= 2 \sum_{x=k+1}^{\infty} (x - \lambda) e^{-\lambda} \cdot \frac{\lambda^x}{x!}, \text{ where } k = [\lambda],$$

$$= 2e^{-\lambda} \sum_{x=k+1}^{\infty} \left[\frac{\lambda^x}{(x-1)!} - \frac{\lambda^{x+1}}{x!} \right] = 2e^{-\lambda} \sum_{x=k+1}^{\infty} [v_x - v_{x+1}]$$

$$= 2e^{-\lambda} \cdot v_{k+1} = 2e^{-\lambda} \cdot \frac{\lambda^{k+1}}{k!} = 2\lambda \cdot \frac{e^{-\lambda} \cdot \lambda^k}{k!}.$$

$$v_x = \frac{\lambda^x}{(x-1)!}$$

FITTING OF POISSON DISTRIBUTION

- ❖ Define the random variable X.
- ❖ Write down the probability – mass function of X .

$$f(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}; x = 0, 1, 2, \dots; \lambda > 0$$

- ❖ Since, the parameter is unknown, so, it has to be estimated by the method of moments.

$$\therefore \hat{\lambda} = \bar{x} = \frac{\sum xf}{\sum f}.$$

$$f(0) = \frac{e^{-\hat{\lambda}} \cdot \hat{\lambda}^0}{0!} = e^{-\hat{\lambda}}.$$

- ❖ Recursion relation is $f(x) = \frac{\hat{\lambda}}{x} \cdot f(x-1); x = 1, 2, \dots$
- ❖ Expected frequencies can be calculated by using the following columns:

x (1)	$\frac{\hat{\lambda}}{x}$ (2)	$f(x) = \frac{\hat{\lambda}}{x} f(x-1)$ (3)	Expected frequency = Nxf(x) (4)	Observed frequency (5)
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- ❖ Probability corresponding to the last row is [1 - total probability up to the previous row of the last row]

Problem:

If $\frac{f(x)}{f(x-1)} = \frac{\lambda}{x}$, $x = 1, 2, \dots$ then find the p.m.f. of X .

Solution:

$$\therefore \frac{f(x)}{f(x-1)} = \frac{\lambda}{x} \text{ so, } f(x) = \frac{\lambda}{x} f(x-1), x = 1, 2, 3, \dots$$

$$\text{Now, } f(1) = \frac{\lambda}{1} f(0) = \frac{\lambda}{1!} f(0)$$

$$f(2) = \frac{\lambda}{2} f(1) = \frac{\lambda}{2} \cdot \frac{\lambda}{1} f(0) = \frac{\lambda^2}{2!} f(0)$$

$$f(3) = \frac{\lambda}{3} f(2) = \frac{\lambda}{3} \cdot \frac{\lambda^2}{2!} f(0) = \frac{\lambda^3}{3!} f(0)$$

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$$\because f(x) \text{ is p.m.f.}, \text{ so } \sum_{x=0}^{\infty} f(x) = 1$$

$$\therefore f(0) + f(1) + f(2) + f(3) + \dots = 1$$

$$Or, f(0) \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right] = 1$$

$$Or, f(0) \cdot e^\lambda = 1$$

$$Or, f(0) = e^{-\lambda}$$

$$f(1) = e^{-\lambda} \cdot \frac{\lambda}{1!}, \quad f(2) = e^{-\lambda} \cdot \frac{\lambda^2}{2!}, \quad f(3) = e^{-\lambda} \cdot \frac{\lambda^3}{3!}$$

$$and \ so \ on. \quad \therefore f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

Problem:

A random variable X follows Poisson distribution with parameter m .

Show that $P(X \text{ is even}) = \frac{1}{2}(1 + e^{-2m})$.

Proof:

$$P(X = x) = e^{-m} \frac{m^x}{x!}, x = 0, 1, 2, \dots$$
$$= 0, \quad \text{otherwise}$$

$$P(X \text{ is even}) = P(X = 0) + P(X = 2) + P(X = 4) + \dots$$

$$= e^{-m} \left[\frac{m^0}{0!} + \frac{m^2}{2!} + \frac{m^4}{4!} + \dots \right]$$

$$e^m = 1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots$$

$$e^{-m} = 1 - m + \frac{m^2}{2!} - \frac{m^3}{3!} + \dots$$

$$e^{-m} + e^m = 2\left(1 + \frac{m^2}{2!} + \frac{m^4}{4!} + \dots\right)$$

$$\therefore P(X \text{ is even}) = e^{-m} \cdot \frac{e^{-m} + e^m}{2} = \frac{1}{2}(1 + e^{-2m}).$$

Problem:

If X and Y are independent Poisson variables such that

$P(X = 2) = P(X = 3)$ and $P(Y = 4) = P(Y = 5)$,

find the standard deviation of $2X - Y$.

Solution:

Let, $P(X = x) = e^{-m} \frac{m^x}{x!}$, $x = 0, 1, 2, \dots$
 $= 0$, otherwise

Let, $P(Y = y) = e^{-n} \frac{n^y}{y!}$, $y = 0, 1, 2, \dots$

$= 0$, otherwise
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$$P(X = 2) = P(X = 3)$$

$$Or, \frac{e^{-m} m^2}{2!} = \frac{e^{-m} m^3}{3!} \quad Or, m = 3$$

$$P(Y = 4) = P(Y = 5)$$

$$Or, \frac{e^{-n} n^4}{4!} = \frac{e^{-n} n^5}{5!} \quad Or, n = 5$$

$$Var(2X - Y) = 4Var(X) + Var(Y) = 4 \cdot 3 + 5 = 17$$

Standard deviation of $(2X - Y)$ = $\sqrt{17}$.

Problem:

If a poisson distribution has double modes at 2 and 3, what is the probability that the variable will take either 2 or 3?

Solution:

Let $X \sim P(\lambda)$ and since it has double modes, so

$$P(X = 2) = P(X = 3) \text{ or, } \frac{e^{-\lambda} \cdot \lambda^2}{2!} = \frac{e^{-\lambda} \cdot \lambda^3}{3!}$$

or, $\lambda = 3$.

$$P(X = 2 \cup X = 3) = e^{-3} \cdot \left[\frac{3^2}{2!} + \frac{3^3}{3!} \right] = 9 \cdot e^{-3}.$$

Problem:

a) If $P(X = 3) = P(X = 4)$ for a Poisson random variable X , then find

i) the mean of the distribution.

ii) $P(X = 0)$ iii) $P(1 \leq X \leq 3)$.

b) If $2.P(X = 0) = P(X = 1)$ for a Poisson random variable X , then find

$P(X > 1 / X < 3)$.

Solution:

a) Let $X \sim P(\lambda)$.

$$P(X = 3) = P(X = 4) \quad \text{or}, \frac{e^{-\lambda} \cdot \lambda^3}{3!} = \frac{e^{-\lambda} \cdot \lambda^4}{4!}$$

$$\text{or, } \lambda = 4$$

i) the mean of the distribution is 4.

ii) $P(X = 0) = e^{-4} = 0.02$.

iii) $P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$.

$$= e^{-4} \left[\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} \right] = 0.453$$

$$b) 2.P(X = 0) = P(X = 1) \text{ or, } 2 \cdot \frac{e^{-\lambda} \cdot \lambda^0}{0!} = \frac{e^{-\lambda} \cdot \lambda^1}{1!}$$

or, $\lambda = 2$.

$$\therefore P(X > 1 / X < 3) = \frac{P(X > 1 \cap X < 3)}{P(X < 3)}.$$

$$= \frac{P(X = 2)}{P(X = 0) + P(X = 1) + P(X = 2)}.$$

$$= \frac{\frac{2^2}{2!}}{\frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!}} = \frac{2}{5}.$$

Problem:

Show that the function

$$f(x) = \frac{e^{-m} \cdot m^x}{(1 - e^{-m}) \cdot x!}, \quad 0 < m < \infty, \quad x = 1, 2, \dots, \infty$$
$$= 0, \text{ otherwise}$$

represents a p.m.f.

Solution:

$$\begin{aligned} & \sum_{x=1}^{\infty} \frac{e^{-m} \cdot m^x}{(1 - e^{-m}) \cdot x!} \\ &= \frac{e^{-m}}{(1 - e^{-m})} \sum_{x=1}^{\infty} \frac{m^x}{x!} \\ &= \frac{e^{-m}}{(1 - e^{-m})} \left[\frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \dots \dots \right] \\ &= \frac{e^{-m} \cdot (e^m - 1)}{(1 - e^{-m})} = 1 \end{aligned}$$

So, given function is a p.m.f.

Thank you.....