

# **NORMAL OR GAUSSIAN DISTRIBUTION**

**Presented  
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**Definition:**

A continuous random variable  $X$  is said to have a normal distribution if its probability density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \\ \sigma > 0 \\ = 0, \text{ otherwise}$$

If  $X$  follows normal distribution with parameters  $\mu$  and  $\sigma^2$ , then it is denoted by  $N(\mu, \sigma^2)$ .

## Standard normal variable:

If  $X$  follows  $N(\mu, \sigma^2)$  then  $Z = \frac{X - \mu}{\sigma}$  is said to have a standard normal distribution if its p.d.f. is defined by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad -\infty < z < \infty$$
$$= 0, \text{ otherwise}$$

*Here,  $Z \sim N(0,1)$ .*

## Distribution function of a Standard normal variable:

*The distribution function  $\Phi(z)$  of a standard normal variate,  $Z$  is defined by*

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^z \phi(t) dt = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

# Properties of $\Phi(z)$ :

$$i) \Phi(-z) = 1 - \Phi(z)$$

$$\begin{aligned} ii) P(a \leq X \leq b) &= P\left(\frac{a - \mu}{\sigma} \leq \frac{X - \mu}{\sigma} \leq \frac{b - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \end{aligned}$$

$$iii) P(Z \leq a) = P(Z \geq -a)$$

## Mean of normal distribution:

Since,  $X \sim N(\mu, \sigma^2)$  so,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty,$$

$$\sigma > 0$$

$= 0$  , otherwise

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} (\mu + \sigma.z) \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot \sigma \cdot dz$$

$$\text{Let, } \frac{x - \mu}{\sigma} = z;$$

$$\therefore dx = \sigma dz$$

$$= \mu \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + \sigma \cdot \int_{-\infty}^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= 2 \cdot \mu \cdot \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$\text{Let, } \frac{z^2}{2} = u; z = \sqrt{2u}$$

$$\therefore z \cdot dz = du$$

$$= 2.\mu.\int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u} \cdot \frac{du}{\sqrt{2u}}$$

$$= \mu.\frac{1}{\Gamma(\frac{1}{2})} \int_0^{\infty} e^{-u} .u^{\frac{1}{2}-1} .du$$

$$= \mu.\frac{1}{\Gamma(\frac{1}{2})} .\Gamma(\frac{1}{2}) = \mu.$$

## Variance of normal distribution:

$$E(X - \mu)^2 = \int_{-\infty}^{\infty} (x - \mu)^2 \cdot f(x) dx$$

$$= \int_{-\infty}^{\infty} (x - \mu)^2 \cdot \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{x - \mu}{\sigma}\right)^2} dx$$

$$\text{Let, } \frac{x - \mu}{\sigma} = z;$$

$$\therefore dx = \sigma dz$$

$$= \int_{-\infty}^{\infty} (\sigma \cdot z)^2 \cdot \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} z^2} \cdot \sigma dz$$

$$= 2.\sigma^2 \int_0^{\infty} z^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$\text{Let, } \frac{z^2}{2} = u ; z = \sqrt{2u} \\ \therefore z \cdot dz = du$$

$$= 2.\sigma^2 \int_0^{\infty} 2u \cdot \frac{1}{\sqrt{2\pi}} e^{-u} \frac{du}{\sqrt{2u}}$$

$$= 2\sigma^2 \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^{\infty} u^{1-\frac{1}{2}} \cdot e^{-u} \cdot du$$

$$= 2\sigma^2 \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^{\infty} u^{\frac{3}{2}-1} \cdot e^{-u} \cdot du$$

$$= 2\sigma^2 \cdot \frac{1}{\Gamma\left(\frac{1}{2}\right)} \cdot \Gamma\left(\frac{3}{2}\right) = 2\sigma^2 \cdot \frac{1}{2} = \sigma^2.$$

*Since,  $\Gamma \lambda = (\lambda - 1)\Gamma(\lambda - 1)$*

$$\text{so, } \Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2} - 1\right)\Gamma\left(\frac{3}{2} - 1\right).$$

## Mean deviation of normal distribution:

$$MD_{\mu} = E[|X - \mu|] = \int_{-\infty}^{\infty} |x - \mu| \cdot f(x) dx$$

$$= \int_{-\infty}^{\infty} |x - \mu| \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{Let, } \frac{x - \mu}{\sigma} = z$$

$$\therefore dx = \sigma dz$$

$$= \sigma \int_{-\infty}^{\infty} |z| \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot \sigma dz$$

$$= 2\sigma \int_0^{\infty} z \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$

$$= 2\sigma \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u} du$$

Let,  $\frac{z^2}{2} = u$  ;  $z = \sqrt{2u}$   
 $\therefore z \cdot dz = du$

$$= 2\sigma \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u} \cdot u^{1-1} du$$

$$= \sqrt{\frac{2}{\pi}} \cdot \sigma = \frac{4}{5} \sigma (\text{approximately})$$

*If  $Z \sim N(0,1)$  then mean deviation about mean*

*is  $\sqrt{\frac{2}{\pi}}$ . Since,  $\sigma = 1$ .*

## Median of normal distribution:

Let  $X \sim N(\mu, \sigma^2)$  and median be  $M$ .

$$\int_{-\infty}^M f(x) dx = \frac{1}{2}$$

$$\text{Or, } \int_{-\infty}^M \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \frac{1}{2}$$

$$\text{Let, } \frac{x-\mu}{\sigma} = z;$$

$$\therefore dx = \sigma dz$$

$$\text{Or, } \int_{-\infty}^{\frac{M-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \frac{1}{2} = \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$\text{So, } \frac{M-\mu}{\sigma} = 0; \therefore \text{Median } (M) = \mu.$$

## Mode of normal distribution:

Since,  $X \sim N(\mu, \sigma^2)$  so,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, \quad -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$$
$$= 0, \text{ otherwise}$$

$$f'(x) = \frac{df(x)}{dx} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \cdot \left[ -\frac{1}{2} \cdot 2 \left( \frac{x-\mu}{\sigma} \right) \cdot \frac{1}{\sigma} \right]$$
$$= f(x) \cdot \left[ -\frac{1}{2} \cdot 2 \left( \frac{x-\mu}{\sigma} \right) \cdot \frac{1}{\sigma} \right] = -f(x) \cdot \frac{x-\mu}{\sigma^2}.$$

$$f'(x) = 0 \Rightarrow x - \mu = 0 \Rightarrow x = \mu.$$

$$f''(x) = f(x) \cdot \left( \frac{x - \mu}{\sigma^2} \right)^2 - f(x) \cdot \frac{1}{\sigma^2}.$$

$$\left[ f''(x) \right]_{x=\mu} = \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left[ \frac{x-\mu}{\sigma} \right]^2} \cdot \left( \frac{x-\mu}{\sigma^2} \right)^2 \right]_{x=\mu}$$

$$- \left[ \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left[ \frac{x-\mu}{\sigma} \right]^2} \cdot \frac{1}{\sigma^2} \right]_{x=\mu}.$$

$$= - \left[ \frac{1}{\sigma^3 \sqrt{2\pi}} \right] < 0.$$

*Hence,  $x = \mu$ , is the mode of the distribution.*

## Points of inflexion of the normal distribution:

Points of inflexion are those points for which

$$f''(x) = 0 \text{ and } f'''(x) \neq 0.$$

$$\text{So, } f''(x) = f(x) \cdot \left( \frac{x - \mu}{\sigma^2} \right)^2 - f(x) \cdot \frac{1}{\sigma^2} = 0.$$

$$\text{or, } \left( \frac{x - \mu}{\sigma^2} \right)^2 = \frac{1}{\sigma^2}. \quad \text{or, } x = \mu \pm \sigma.$$

$$f'''(x) = -f(x) \cdot \left(\frac{x-\mu}{\sigma^2}\right)^3 + f(x) \cdot 2 \cdot \frac{x-\mu}{\sigma^4} + f(x) \cdot \frac{1}{\sigma^2} \cdot \frac{x-\mu}{\sigma^2}.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2} \left[ -\left(\frac{x-\mu}{\sigma^2}\right)^3 + 3 \cdot \frac{x-\mu}{\sigma^4} \right].$$

$$\left[ f'''(x) \right]_{x=\mu \pm \sigma} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}} \left[ \mp \frac{1}{\sigma^3} \pm \frac{3}{\sigma^3} \right].$$

$$= \pm \frac{2}{\sigma^4\sqrt{2\pi}} e^{-\frac{1}{2}}.$$

# Even Order Central Moments

$$\mu_{2r} = E(X - \mu)^{2r} = \int_{-\infty}^{\infty} (x - \mu)^{2r} f(x) dx$$

$$= \int_{-\infty}^{\infty} (x - \mu)^{2r} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\text{Let, } \frac{x - \mu}{\sigma} = z$$

$$\therefore dx = \sigma dz$$

$$= \int_{-\infty}^{\infty} (\sigma \cdot z)^{2r} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \cdot \sigma \cdot dz$$

$$\text{Let, } \frac{z^2}{2} = u ; z = \sqrt{2u}$$

$$= \sigma^{2r} \cdot 2 \cdot \int_0^{\infty} (2 \cdot u)^r \cdot \frac{1}{\sqrt{2\pi}} e^{-u} \cdot \frac{du}{\sqrt{2u}}$$

$$\therefore z \cdot dz = du$$

$$= \sigma^{2r} \cdot 2^r \cdot \int_0^{\infty} (u)^{r-\frac{1}{2}} \cdot \frac{1}{\sqrt{\pi}} e^{-u} \cdot du$$

$$= \sigma^{2r} \cdot 2^r \cdot \frac{1}{\sqrt{\pi}} \cdot \int_0^{\infty} (u)^{r+\frac{1}{2}-1} \cdot e^{-u} \cdot du$$

$$= \sigma^{2r} \cdot 2^r \cdot \frac{1}{\sqrt{\pi}} \cdot \Gamma\left(r + \frac{1}{2}\right)$$

$$= \sigma^{2r} \cdot 2^r \cdot \frac{1}{\sqrt{\pi}} \cdot \left(r - \frac{1}{2}\right) \cdot \left(r - \frac{3}{2}\right) \cdot \dots \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right)$$

$$= \sigma^{2r} \cdot (2r-1) \cdot (2r-3) \cdot \dots \cdot 1 \cdot$$

$$\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

# Odd Order Central Moments

$$\mu_{2r+1} = E(X - \mu)^{2r+1} = \int_{-\infty}^{\infty} (x - \mu)^{2r+1} f(x) dx$$

$$= \int_{-\infty}^{\infty} (x - \mu)^{2r+1} \cdot \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} dx$$

$$= \int_{-\infty}^{\infty} (\sigma \cdot z)^{2r+1} \cdot \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(z)^2} \cdot \sigma \cdot dz$$

$$= 0$$

*$\therefore$  integrand is an odd function in  $z$ .*

$$\text{Let, } \frac{x - \mu}{\sigma} = z$$

$$\therefore dx = \sigma dz$$

# Recursion Relation for Moments

$$\mu_{2r} = E(X - \mu)^{2r} = \int_{-\infty}^{\infty} (x - \mu)^{2r} f(x) dx$$

$$= \int_{-\infty}^{\infty} (x - \mu)^{2r} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$\frac{d\mu_{2r}}{d\sigma} = \int_{-\infty}^{\infty} (x - \mu)^{2r} \cdot \frac{1}{\sqrt{2\pi}} \cdot \left\{ -\frac{1}{\sigma^2} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \right\} dx$$

$$+ \int_{-\infty}^{\infty} (x - \mu)^{2r} \cdot \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \frac{(x - \mu)^2}{\sigma^3} dx$$

$$= -\frac{1}{\sigma} \cdot \mu_{2r} + \frac{1}{\sigma^3} \cdot \mu_{2r+2}$$

$$\therefore \mu_{2r+2} = \sigma^3 \left[ \frac{1}{\sigma} \cdot \mu_{2r} + \frac{d\mu_{2r}}{d\sigma} \right] \quad ; r = 0, 1, \dots$$

$$\mu_2 = \sigma^3 \left[ \frac{1}{\sigma} \cdot \mu_0 + \frac{d\mu_0}{d\sigma} \right] = \sigma^2$$

$$\mu_4 = \sigma^3 \left[ \frac{1}{\sigma} \cdot \mu_2 + \frac{d\mu_2}{d\sigma} \right] = \sigma^3 \left[ \frac{1}{\sigma} \cdot \sigma^2 + \frac{d\sigma^2}{d\sigma} \right] = 3\sigma^4$$

# Quartile Deviation

$$P(X \leq Q_3) = \int_{-\infty}^{Q_3} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\frac{Q_3-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \Phi\left(\frac{Q_3 - \mu}{\sigma}\right)$$

$$\text{Let, } \frac{x - \mu}{\sigma} = z$$

$$\therefore dx = \sigma dz$$

$$\text{Similarly, } P(X \leq Q_1) = \Phi\left(\frac{Q_1 - \mu}{\sigma}\right)$$

$$\Phi\left(\frac{Q_3 - \mu}{\sigma}\right) = \frac{3}{4} = 0.75 = \Phi(0.67)$$

$$\therefore Q_3 = \mu + 0.67\sigma$$

$$\Phi\left(\frac{Q_1 - \mu}{\sigma}\right) = \frac{1}{4} = 1 - 0.75 = 1 - \Phi(0.67) = \Phi(-0.67)$$

$$\therefore Q_1 = \mu - 0.67\sigma$$

$$Q.D. = \frac{Q_3 - Q_1}{2} = \frac{\mu + 0.67\sigma - \mu - 0.67\sigma}{2} = 0.67\sigma$$

## Problem :

The p.d.f. of a certain random variable  $X$  is

given by  $f(x) = \sqrt{\frac{2}{\pi}} \cdot e^{-2(x-3)^2}$ ,  $-\infty < x < \infty$

Identify the distribution of  $X$  and find the mean and variance.

**Solution :**

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot e^{-2(x-3)^2} = \frac{1}{\frac{1}{2} \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-3}{\frac{1}{2}}\right)^2}.$$

*Comparing with the normal p.d.f.*

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \text{ it is found that}$$

$X \sim N\left(3, \frac{1}{4}\right)$ . Here, mean  $\mu = 3$  and variance

$$\sigma^2 = 1/4.$$

## *Problem :*

*Let  $X \sim N(0,1)$  and  $Q_1, Q_3$  be, respectively the 1st and 3rd quartiles of this distribution.*

*A random variable  $Y$  is defined as*

$$\begin{aligned} Y &= 1, \text{ if } X < Q_1 \\ &= 2, \text{ if } Q_1 \leq X < Q_3 \\ &= 3, \text{ if } X \geq Q_3 \end{aligned}$$

*Compute variance of  $Y$ .*

*Solution :* Here  $X \sim N(0,1)$ .

$\therefore Y = 1$ , with probability if  $1/4$   
 $= 2$ , with probability if  $1/2$   
 $= 3$ , with probability if  $1/4$

$$E(Y) = \sum_y y.P(Y = y) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 3 \cdot \frac{1}{4} = 2$$

$$E(Y^2) = \sum_y y^2.P(Y = y) = 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{4} = \frac{18}{4}$$

$$V(Y) = E(Y^2) - E^2(Y) = 18/4 - 4 = 1/2.$$

## Problem :

If  $X$  and  $Y$  are independent random

variables with p.d.f.  $g(x) = \frac{e^{-\frac{1}{18}(x-7)^2}}{\sqrt{18\pi}}$

and  $h(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$ , respectively, then find

out the s.d. of  $(3X - 3Y)$ .

*Solution :*

$$g(x) = \frac{e^{-\frac{1}{18}(x-7)^2}}{\sqrt{18\pi}} = \frac{e^{-\frac{1}{2}\left(\frac{x-7}{3}\right)^2}}{3\sqrt{2\pi}}$$

*Comparing with the normal p.d.f. it is found that  $X \sim N(7,9)$  and  $Y \sim N(0,1)$ .*

$$\text{Var}(3X - 3Y) = 9\text{Var}(X) + 9\text{Var}(Y) = 9.9 + 9.1 = 90.$$

$$\text{s.d.}(3X - 3Y) = 3\sqrt{10}$$

*$\text{Cov}(X, Y) = 0$ , since  $X$  and  $Y$  are independent.*

## *Problem :*

*Explain whether the following function can*

*be considered as a p.d.f.  $f(x) = \frac{5.e^{-25.x^2}}{\sqrt{\pi}}$ ,*

*$-\infty < x < \infty$ . If so write down the name,  
mean and variance.*

*Solution :*

$$f(x) = \frac{e^{-\left(\frac{x}{\frac{1}{5}}\right)^2}}{\frac{1}{5} \cdot \sqrt{\pi}}, -\infty < x < \infty$$

*Comparing with the normal p.d.f. it is*

*found that  $X \sim N(0, \frac{1}{25})$ .*

## Problem :

If  $X$  is a normal whose mean, variance and 3rd quartile are 5, 4 and  $\alpha$  respectively, find the value of  $P(X \geq 5 - 2\alpha)$ .

## Solution :

If  $X \sim N(\mu, \sigma^2)$  then 3rd quartile is  $\mu - 0.67\sigma$ .

$$\therefore \alpha = 5 - 0.67(2) = 3.66$$

$$P(X \geq 5 - 2\alpha) = P\left(\frac{X - 5}{2} \geq \frac{5 - 2\alpha - 5}{2}\right)$$

$$= P(Z \geq -\alpha) = P(Z \leq \alpha) = P(Z \leq 3.66)$$

*Since, it is known that*

$$P(-3 \leq Z \leq 3) = 0.9973,$$

*$\therefore P(Z \leq 3.66)$  is nearest to 1.*

**Thank You**

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